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Compatibility of systems of super differential equations*

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Abstract. *A precise definition of the compatibility of systems of super differential equations is given, called Gröbner integrability, which includes the involutiveness as a special case. Several practical criterions of Gröbner integrability are given, one of which generalizes the formal part of the Cartan-Kähler theorem.*

INTRODUCTION

1. Recently in the theory of supergravity field equations are written down using the so called superspace formalism: fields are superfunctions on a superspace and the equations are systems of super differential equations. Although many concrete systems of super differential equations have been already deeply analyzed, there seems to be no general theory comparable to the formal theory of systems of differential equations, which is by now a completed theory both theoretically and practically.

2. The remarkable novelty of systems of super differential equations is that it has more compatibility conditions than usual systems of differential equations. For example even a single super differential equation has nontrivial

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compatibility conditions and hence can be incompatible. This fact is actually one of the essential power of the concept of systems of super differential equations. For example Witten ([7]) uses this fact to express solutions of the Yang-Mills equations as certain vector bundles over super spaces.

3. In this paper we consider systems of super differential equations, and give several criterions for their compatibility and when they are compatible give a method to describe the space of formal solutions.

4. The key concept is the formal Gröbner integrability of systems of super differential equations.

The introduction of this notion is strongly influenced by the notion of the Gröbner basis in the computer algebra theory (cf. [1] for example). In fact the analogue of the Gröbner basis for the basis of differential ideals in differential super polynomial algebra is exactly the formal Gröbner integrability of systems of super differential equations. Thus we might as well say that a system of super differential equations constitute a Gröbner basis of the differential ideal generated by it instead of saying that it is formally Gröbner integrable.

When a system of super differential equations is formally Gröbner integrable, we know all about that system, in other words, there are no unexpected consequences of it. Thus for example it is easy to judge whether a differential relation among the unknown functions is a consequence of that system. This allows theoretically automated theorem proving in the theory of super differential geometry in the line of [8]. Further we can also solve the super differential elimination problem easily.

We should remark here that the formal Gröbner integrability of systems of super differential equations depends on the order introduced on the set of partial derivatives, and hence is not a concept invariantly defined. However this is rather an advantage when we study concrete systems of super differential equations. For example when we want to solve a super differential elimination problem we can and must choose an appropriate order.

The framework of the formal geometry in the sense of Gelfand (cf. [2] for example) is essential for the formulation of the concept of formal Gröbner integrability.

5. Since the formal Gröbner integrability consists of infinite number of conditions, the main problem is to give practical criterions for the formal Gröbner integrability of systems of super differential equations.

The main purpose of the present paper is to give several such criterions, which might be called formal Cartan Kähler theorems.

The first criterion asserts that the formal Gröbner integrability of a system of super differential equations can be established checking only the minimal set of compatibility conditions. When the system consists of the usual differential equations, then this result is essentially the same as given classically by Riquier, Janet, et al. (cf. [4]).

We remark that these results are sufficient for the formal geometric study of concrete super differential equations (cf. [6]) even for the pure even case.

The second one is a more refined one, based on the notion of the involutiveness of systems super differential equations. This criterion is very close to the usual Cartan Kähler theorem except that the notion of the involutiveness adopted in this paper depends on the choice of coordinates. I have not yet been able to rephrase this involutiveness as the acyclicity of the super Koszul complex associated with the system.

6. The prolongation theorem in the sense of Cartan-Kuranish is not treated in this paper. However we remark that a naive version of it is rather obvious: when a given system of super differéntial equations is not formally Gröbner integrable, we add to it the new equations which are obtained by checking the minimal compatibility conditions until the system is formally Gröbner integrable. When adding the new equations does not destroy the regularity, it is easy to see that this process terminates in a finite number of steps.

7. We comment here on the two simplifications assumed in this paper.

We consider only systems of super differential equations on superdomains. This enables us not only to concentrate on the essential features of the problem of the compatibility but also to avoid the difficult choice of various formulation of supermanifolds. It doses not seem difficult to generalize the results of this paper in global setting, once one has chosen a formulation of supermanifold.

Secondly we consider only regular systems of super differential equations. This is inevitable in order to avoid the various difficult questions involving differential ideals, which however should be taken up some day. On the other hand this restriction is not so inappropriate in the application point of view since most of the systems encountered in physics are regular.

8. The outline of this paper is as follows: in section one, we fix notations of super multi-indices, which play important role throughout this paper.

In section two, we review basic concepts on super spaces necessary for later parts. We consider general superfunctions witch coefficients in arbitrary superalgebra, which seems to be necessary to treat general super differential equation. However the reader can assume the coefficients super algebra G to be the trivial algebra $\mathbb{R}$ for the first reading. We remark that the notations used there are slitely unusual but seems to make local arguments quite concise even when confined to the purely even case.

The simple lemmas in the subsection 3.4 give us powerful tools to manipulate regular ideals, which substitute in a sence the difficult delicate arguments necessary for manipulating general ideals.

Section four defines super differential equations in terms of the jet superspace and section five introduce the key concept of formal Gröbner integrability.

Section six gives a few sufficient conditions for the formal Gröber integrability. The subsection 6.1 is crucial for the statement of such conditions.

The final section seven gives a refined version of the sufficient conditions, which is in pure even situation reduced to the formal part of the usual Cartan-Kähler Theorem as formulated by Goldschmidt ([3]).

9. Finally we remark that even in the pure even case, our approach gives a new simpler method of analyzing concrete systems of differential equations.

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NOTATIONS

$A^a := \{b; b < a\}$, when $(A, <)$ is an ordered set.

$A^{(a)} := A^a \cup \{a\}$, when $(A, <)$ is an ordered set.

$A \amalg B$: the disjoint union of sets A and B.

B^c : the complement of a subset B.

$\delta_{ij} := 1$ if $i = j$, $= 0$ if $i \neq j$.

$|I| :=$ the number of elements of a set I.

$N(n) := \{1, 2, \ldots, n\}$.

$\mathbb{R} :=$ the field of real numbers.

$\mathbb{Z} :=$ the ring of integers.

$\mathbb{Z}_+ := \{n \in \mathbb{Z}; n \geqslant 0\}$.

$\mathbb{Z}_{++} := \{n \in \mathbb{Z}; n > 0\}$.

$\mathbb{Z}_+^* := \mathbb{Z}_+ \amalg \{\infty\}$.

$\mathbb{Z}_2 := \{\bar{0}, \bar{1}\} = \mathbb{Z}/2\mathbb{Z}$.

More specialized notations and terminologies frequently used are listed below according to the place where they are first introduced (A and I stand for $\mathbb{Z}_2$-sets and G for a superalgebra):

1.1. $\mathbb{Z}_2$-set $N(m \mid n)$.

1.2. $sw_r(A), \overline{sw}(A)$.

1.3. $m(A, I)$, admissible order, Z_a $(a \in A)$.

1.4. $sm_r(A, I), sm_{(r)}(A, I), \rho_r(\Xi)$ $(\Xi \subset sm(A, I))$, $U \boxplus V$.

2.1. augmentation, augmented superalgebra, admissible augmented superalgebra, G-superalgebra

2.2. G-supermodule, ideal of G, $G.X$.

2.3. pi(W), pt(W) (W: a submodule of a free G-supermodule), regular subspace of a free G-supermodule.

2.4. $A(G)$ (A : a $\mathbb{Z}_2$-set), G-point.

2.5. $\mathbb{R}[A], \mathbb{R}_r[A], \mathbb{R}[[A]], P(G): A(G) \to G$ $(P \in \mathbb{R}[A])$.

2.6. $F(A), F_G(A), F_G(A^U)$.

2.7. $\partial_a = \partial/\partial Z_a : F_G(A) \to F_G(A)$ $(a \in A)$.

2.8. $\tau : F_G(A) \to F_G(A_{\bar{0}}) \otimes \mathbb{R}[[A]]$.

2.9. $\mathrm{ev}_\zeta : F_G(A) \to G$ $(\zeta \in A(G))$.

3.1. $F_G(A, I)$, a smooth supermap with coefficients in G from A to I.

3.2. $F^* : F_G(I) \to F_G(A)$ $(F \in F_G(A, I))$.

3.3. superdiffeomorphism.

3.4. $F_G^0(A, B), F_G^{00}(A, B)$ $(B \subset A)$, regular subset of $F_G(A)$,

$E(\varphi)$ $(\varphi \in F^0_G(A, B))$, pi$(E)$, regular ideals.

4.1. $J_r = J_r(A, I)$, $J_\infty = J_\infty(A, I)$, $\mathscr{F}_r = F(J_r)$, u_W $(W \in \overline{sw}(A, I))$,

$j : F(A, I) \to F(A, J(A, I))$ $(B \subset I)$.

4.2. a system of super differential equation, solution, Sol$(\mathscr{E})$, $\mathscr{E}_1 \sim \mathscr{E}_2$.

4.3. regular system of super differential equations (φ) $(\varphi \in F^0(J, \Xi))$, pi$(\mathscr{E})$ ($\mathscr{E}$: a system of super differential equations).

5.1. $d_a : F(J(A, I)) \to F(J(A, I))$ $(a \in A)$.

5.2. $\rho_n(\mathscr{E})$, formally Gröbner integrable.

6.1. $ml\ t(M)$ $(M \in sm(A, I))$, $ml\ t(A, \Xi)$, $Basis(\Xi)$

$(\Xi \subset sm(A))$, $prim(\Gamma)$, 1-acyclic.

6.2. $\psi_{\overline{M}}$ $(\overline{M} \in \rho_\infty(\Xi))$, precompatible.

6.3. good subset, $c(\mu)$, class of μ, $A(\mu)$.

7.1. involutive

7.2. σ_r, $\overline{\sigma}_r$, $Symb_r(\mathscr{E})$.

§1. SUPERINDICES

1.1. $\mathbb{Z}_2$-sets

Let A be a set. A map $p : A \to \mathbb{Z}_2$ is called a $\mathbb{Z}_2$-grading of A. A set endowed with a $\mathbb{Z}_2$-grading is called a $\mathbb{Z}_2$-set. Let (A, p) be a $\mathbb{Z}_2$-set. We put $A_k = p^{-1}(k)$ $(k \in \mathbb{Z}_2)$. We often write $\tilde{a} = p(a)$.

A subset B of a $\mathbb{Z}_2$-set (A, p) is considered as a $\mathbb{Z}_2$-set by the $\mathbb{Z}_2$-grading $p|_B$.

If (A_1, p_1) and (A_2, p_2) are $\mathbb{Z}_2$-sets, we consider the product $A_1 \times A_2$ and the disjoint union $A_1 \amalg A_2$ as $\mathbb{Z}_2$-sets by the $\mathbb{Z}_2$-gradings defined respectively by

$$p((a_1, a_2)) = p_1(a_1) + p_2(a_2), \text{ for } a_i \in A_i \ (i = 1, 2)$$

and

$$p|_{A_i} = p_i \quad \text{for} \quad i = 1, 2.$$

The set $N(m)$ will be regarded as a $\mathbb{Z}_2$-set by

$$N(m)_{\bar{0}} = N(m), \quad N(m)_{\bar{1}} = \phi .$$

We denote by $\overline{N(n)}$ the $\mathbb{Z}_2$-set defined by

$$\overline{N(n)}_{\bar{1}} = N(n), \quad \overline{N(n)}_{\bar{0}} = \phi$$

and put

$$N(m \mid n) := N(m) \amalg \overline{N(n)}.$$

From now on we fix a finite $\mathbb{Z}_2$-set (A, p).

1.2. Set of words

Let A be a $\mathbb{Z}_2$-set. We define

$$w_0(A) := \{\phi\},$$

$$w_r(A) := A^r, \text{ for } r > 0,$$

$$w(A) := \amalg_{r=0}^{\infty} w_r(A).$$

We identify A with $w_1(A)$. The element $(a_1, \ldots, a_r)$ of $w_r(A)$ is denoted simply by $a_1 \ldots a_r$. The juxtaposition defines a multiplication $w(A) \times w(A) \to \to w(A)$. The product of v and w is denoted by vw.

Let $\zeta = \{\zeta_a ; a \in A\}$ be a set of letters. We define

$$\zeta^{\phi} = 1,$$

$$\zeta^{w} := \zeta_{w(1)} \cdots \zeta_{w(r)} \text{ for } w \in w_r(A).$$

Remark (1.2.1). In explicit calculation, it is often confusing to express words on A directly, e.g., when A consists of numbers. In such cases, it is convenient to introduce letters indexed by A, e.g., $\{z_a ; a \in A\}$ and express a word w on A by the corresponding monomial z^w. For example, the word *abbc* is expressed by the monomial $z_a z_b^2 z_c$. ■

For $w = (w(1), \ldots, w(r)) \in w(A)$ and $\kappa \in \mathbb{Z}_2$, define

$$N(w, \kappa) = \{i \in N(r);\ w(i) \in A_k\},$$

$$\ell_{\kappa}(w) := {}^{\#}N(w, \kappa),$$

$$\ell(w) := \ell_{\bar{0}}(w) + \ell_{\bar{1}}(w) = r.$$

Suppose from now on that A is totally ordered.

For $w \in w(A)$ with $\ell_{\bar{1}} := \ell_{\bar{1}}(w) > 0$, we denote by ρ_w the order preserving bijection from $N(\ell_{\bar{1}})$ to $N(w, \bar{1})$.

The permutation group $\mathfrak{S}_r$ acts on $w_r(A)$ from the right by

$$w.\pi := (w(\pi 1), \ldots, w(\pi r)),$$

for $w \in w_r(A), \pi \in \mathfrak{S}_r$. For $(w, \pi) \in w_r(A) \times \mathfrak{S}_r$, define

$$sgn(w, \pi) := \begin{cases} 1 \text{ if } \ell_{\overline{1}}(w) = 0, \\ sgn(\rho_{w.\pi}^{-1} \circ \rho_w) \text{ if } \ell_{\overline{1}}(w) > 0. \end{cases}$$

By definition

$$sgn(w, \pi_1 \pi_2) := sgn(w.\pi_1, \pi_2)\, sgn\,(w, \pi_1),$$

for $w \in w_r(A)$, $\pi_1, \pi_2 \in \mathfrak{S}_r$.

Define $\overline{w}_0(A) := w_0(A)$ and, for $r > 0$,

$$\overline{w}_r(A) := \{ w \in w_r(A);\ w(1) \leqslant \ldots \leqslant w(r)\}.$$

Put

$$\overline{w}(A) := \cup_{r=0}^{\infty} \overline{w}_r(A).$$

Obviously we have the following lemma.

LEMMA (1.2.2).

(i) *The action* $(w, \pi) \to w.\pi$ *induces a surjection from* $\overline{w}_r(A) \times \mathfrak{S}_r$ *to* $w_r(A)$.

(ii) *The condition* $w.\pi = w'.\pi'$ *for* $(w, \pi), (w', \pi') \in \overline{w}_r(A) \times \mathfrak{S}_r$ *implies* $w = w'$ *and* $sgn(w, \pi) = sgn(w', \pi')$. ∎

We define $sw_0(A) = w_0(A)$ and, for $r > 0$,

$$sw_r(A) = \{ w \in w_r(A);\ {}^{\#}\{i; w(i) = a\} \leqslant 1 \text{ for all } a \in A_{\overline{1}}\}.$$

We put

$$sw(A) := \cup_{r=0}^{\infty} sw_r(A),$$

$$\overline{sw}_r(A) := sw_r(A) \cap \overline{w}(A),$$

$$\overline{sw}(A) := sw(A) \cap \overline{w}(A).$$

1.3. Multi-indices

Define

$$m(A) := Map(A, \mathbb{Z}_+).$$

For $\mu \in m(A)$, we put

$$|\mu| := \Sigma_{a \in A} \mu(a),$$
$$\tilde{\mu} := \Sigma_{a \in A_1} \mu(a) \bmod 2,$$
$$\mu! := \Pi_{a \in A} \mu(a)!.$$

We consider $m(A)$ as a $\mathbb{Z}_2$-sets by the $\mathbb{Z}_2$-grading $\mu \to \tilde{\mu}$.
Define

$$m_r(A) := \{\mu \in m(A);\ |\mu| = r\}, \text{ for } r \in \mathbb{Z}_+,$$
$$m_{(r)}(A) := \{\mu \in m(A);\ |\mu| \leqslant r\}, \text{ for } r \in \mathbb{Z}_+^*.$$

For each $a \in A$, we define $\delta_a \in m(A)$ by $\delta_a(b) = \delta_{ab}$.
Define $\mu : w(A) \to m(A)$ by $\mu(\phi) := 0$ and

$$\mu(w) := \Sigma_{i=1}^{\ell} \delta_{w(i)}$$

for $w = w(1) \dots w(\ell)$. Obviously $|\mu(w)| = \ell(w)$.
Suppose A is totally ordered.

LEMMA (1.3.1). *The restriction* $\mu|_{\overline{w}(A)}$ *is a bijection.* ∎

We shall denote the inverse of $\mu|_{\overline{w}(A)}$ by $w : m(A) \to \overline{w}(A)$.
Suppose $\{z_a : a \in A\}$ is a set of letters indexed by A. For $\mu \in m(A)$, we define $z^{\mu} := z^{w(\mu)}$. Note that this depends on the order of A if A_{I} is not empty.

Remark (1.3.2). When A is totally ordered, the bijection w from $m(A)$ to $\overline{w}(A)$ enables us to identify a multi-index μ with the monomial $z^{w(\mu)}$, when $\{z_a, a \in A\}$ is a set of letters (cf. Remark (1.2.1)). For example, suppose $A = \{1, 2, 3\}$ with the standard order and x, y, z are letters corresponding to 1, 2, 3 respectively. Then a multi-index m is identified with the monomial $x^{m(1)} y^{m(2)} z^{m(3)}$. ∎

For $\mu, \mu' \in m(A)$, we define the elements $\mu + \mu'$ of $m(A)$ by

$$(\mu + \mu')(a) := \mu(a) + \mu'(a),$$

for $a \in A$.
Let I be a $\mathbb{Z}_2$-set. Define

$$m(A, I) := m(A) \times I,$$

with the product $\mathbb{Z}_2$-grading. For $M \in m(A, I)$, we write its $m(A)$ – and I – components respectively by $\mu(M)$ and $i(M)$. When I consists of a single even element, we identify $m(A, I)$ with $m(A)$.

We put

$$m_r(A, I) := m_r(A) \times I,$$

$$m_{(r)}(A, I) := m_{(r)}(A) \times I.$$

For $\mu \in m(A)$ and $M \in m(A, I)$, we put

$$\mu + M := (\mu + \mu(M),\ i(M)).$$

For subsets U of $m(A)$ and V of $m(A, I)$, we put

$$U + V := \{u + v : u \in U,\ v \in V\}.$$

Let Ξ be a subset of $m(A, I)$. For $r \in \mathbb{Z}_+^*$, we define

$$\tilde{\rho}_r(\Xi) := m_{(r)}(A) + \Xi.$$

We define a partial order $\ll$ in $m(A, I)$ by

$$M \ll M' \Leftrightarrow i(M) = i(M') \text{ and } M' \in \tilde{\rho}_\infty(M).$$

A linear order « $<$ » in $m(A, I)$ is called *admissible* if it satisfies the following three conditions:

(i) $M \ll M'$ implies $M < M'$,

(ii) $M < M'$ implies $\mu + M < \mu + M'$ for $\mu \in m(A)$,

(iii) every strictly decreasing sequence of elements is of finite length, i.e., « $<$ » is a well order.

Example (1.3.3). Suppose A and I are totally ordered.

(i) The standard linear order of $m(A, I)$ is the pull back of the lexicographical ordering of $\mathbb{Z}_+ \times m(A) \times I$ by the mapping $M \to (|\mu(M)|, \mu(M), i(M))$. Obviously this order is admissible.

For example, suppose $A = \{1, 2, 3\}$ with the standard order and x, y, z are letters corresponding to 1, 2, 3 respectively. Then the sequence of elements of $m(A)$ in the ascending order starts as follows (cf. Remark (1.3.2)).

$$1, x, y, z,$$

$$x^2, xy, xz, y^2, yz, z^2,$$

$$x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3,$$

$$\dots$$

(ii) Let « $<$ » be the order of $m(A, I)$ defined as follows:

$$M < M' \text{ if (a) } |\mu(M)| < |\mu(M')|,$$
$$\text{or (b) } |\mu(M)| = |\mu(M')| \text{ and } \mu(M) > \mu(M'),$$
$$\text{or (c) } \mu(M) = \mu(M') \text{ and } i(M) < i(M').$$

Obviously this order is admissible.

For example, suppose that A is as in (i) but suppose that the letters corresponding to 1, 2, 3 are now z, y, x. Then the ascending sequence starts as follows:

$$1, x, y, z,$$
$$x^2, xy, y^2, xz, yz, z^2,$$
$$x^3, x^2y, xy^2, y^3, x^2z, xyz, y^2z, xz^2, xz^2, yz^2, z^3,$$
$$\dots$$

1.4. Super multi-indices

Let

$$sm(A) := \{m \in m(A); m(a) = 0 \text{ or } 1 \text{ for } a \in A_{\bar{1}}\},$$
$$sm(A, I) := sm(A) \times I.$$

We put

$$sm_r(A) := m_r(A) \cap sm(A),$$
$$sm(A)_{(r)} := m_{(r)}(A) \cap sm(A),$$
$$sm_r(A, I) := m_r(A, I) \cap sm(A, I),$$
$$sm(A, I)_{(r)} := m_{(r)}(A, I) \cap sm(A, I),$$
$$sm(A, I)^{(2)} := m(A, I)^{(2)} \cap sm(A, I).$$

Define for a subset Ξ of $sm(A, I)$ and $r \in \mathbb{Z}_+^*$,

$$\rho_r(\Xi) := \tilde{\rho}_r(\Xi) \cap sm(A, I).$$

The set $\rho_\infty(\Xi)$ will be denoted simply by $\rho(\Xi)$ (cf. Example (6.1.2)).

For subsets U of $sm(A)$ and Ξ of $sm(A, I)$, we write

$$U \boxplus \Xi := (U + \Xi) \cap sm(A, I).$$

Then

$$sm_r(A, I) = sm_r(A) \boxplus sm_0(A, I).$$

A subset Ξ of $sm(A, I)$ is called an *sm-subset* if

$$sm(A) \boxplus \Xi \subset \Xi,$$

i.e., $\rho_\infty(\Xi) \subset \Xi$.

Let $w : m(A) \to \bar{w}(A)$ be the bijection defined in the previous section. Obviously we have

LEMMA (1.4.1). *The restriction*

$$w|_{sm(A)} : sm(A) \to s\bar{w}(A)$$

is bijective. ∎

§2. SUPERFUNCTIONS

2.1. Superalgebras

A $\mathbb{Z}_2$-graded $\mathbb{R}$-algebra $G = G_0 \oplus G_1$ endowed with 1 is called a *superalgebra* if

$$(1) \qquad ab = (-1)^{\tilde{a}\tilde{b}} ba$$

holds for homogeneous a and b. Here $\tilde{a} = \kappa$ for $a \in A_\kappa$. The real field $\mathbb{R}$ will be considered as a superalgebra with $\mathbb{R}_{\bar{0}} = \mathbb{R}$ and $\mathbb{R}_{\bar{1}} = (0)$. The G_κ-component of a is denoted by a_κ $(\kappa \in \mathbb{Z}_2)$.

A $\mathbb{Z}$-graded algebra $G = \oplus_{i \in \mathbb{Z}_+} G_i$, with 1 and satisfying (1), where $\tilde{a} = i \mod 2$ for $a \in G_i$, will be considered as a superalgebra by

$$G_\kappa := \Sigma_{i \bmod 2 = \kappa} G_i \qquad (\kappa \in \mathbb{Z}_2).$$

Let G and G' be superalgebras. A homomorphism $\varphi : G \to G'$ is an algebra map such that $\varphi(1) = 1$ and $\varphi(G_\kappa) \subset G'_\kappa$ $(\kappa \in \mathbb{Z}_2)$. A homomorphism $G \to \mathbb{R}$ is called *an augmentation of G*. The pair (G, ϵ) is called an *augmentated superalgebra*. An augmented superalgebra (G, ϵ) is called *admissible* if $(\mathrm{Ker}\ \epsilon)^N = 0$ for some $N \in \mathbb{Z}_+$.

Let G' and G'' be superalgebras. The $\mathbb{Z}_2$-graded algebra G defined by $G_\kappa = \oplus_{\kappa' + \kappa'' = \kappa} G'_{\kappa'} \otimes G''_{\kappa''}$ with the product

$$(a' \otimes a'')(b' \otimes b'') = (-1)^{\tilde{a}''\tilde{b}'} a'b' \otimes a''b''$$

is a superalgebra, called *the tensor product of G' and G''*.

A superalgebra G' endowed with a homomorphism $\varphi : G \to G'$ is called a *G-superalgebra*.

2.2. Supermodules

Let G be a superalgebra. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded left G-module. The algebra G acts on V also from the right by $v \cdot g = (-1)^{\tilde{v}\tilde{g}} g\, v$. We say simply V is a G-supermodule.

Let G' and G'' be superalgebras. Let V be a G' supermodule. Then $V \otimes_{\mathbb{R}} G''$ is a $G'' \otimes G'$-supermodule by the action

$$(f \otimes g) \cdot (v \otimes h) := (-1)^{\tilde{g}\tilde{v}} f \cdot v \otimes gh$$

($f \in G'$, $g, h \in G''$, $v \in V$). Similarly $G'' \otimes_{\mathbb{R}} V$ is a $G'' \otimes G'$-supermodule.

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded $\mathbb{R}$-vector space. Then for a superalgebra G, the tensor product $G \otimes V$ is a G-supermodule, called *a free G-supermodule of rank*$(r_0 \mid r_1)$, where $r_\kappa := \dim_{\mathbb{R}} V_\kappa$ ($\kappa \in \mathbb{Z}_2$).

The superalgebra G is itself a G-supermodule. Its G-submodules are called *ideals of G*. For a subset X of G, the ideal generated by X is denoted by $G.X$.

2.3. Free supermodules with ordered bases

Let G be a superalgebra and V a supermodule over G

For a subset U of V, the G-submodule generated by U is denoted by $G.U$. Let I be a $\mathbb{Z}_2$-set with a linear order « $<$ » and

$$V = \oplus_{i \in I} G.e_i.$$

For $v = \Sigma\, v_i e_i \in V$, we put

$$cf(i, v) := v_i,$$

$$pi(v) := \max\{i;\, v_i \neq 0\},$$

$$pt(v) := v_i e_i \qquad (i = pi(v)).$$

Let W be a G-submodule. We put

$$pi(W) := \{pi(w);\ w \in W\} \subset I$$

$$pt(W) := G.\{pt(w);\ w \in W\}.$$

We say W *regular* if there exists

$$\{w_i;\ i \in pi(W)\} \subset W$$

such that

$$pt(w_i) = e_i.$$

2.4. Superspaces

Let A be a $\mathbb{Z}_2$-set. For a superalgebra G, we define a set $A(G)$ by

$$A(G) := \{\zeta : A \to G;\quad \zeta(a) \in G_{\tilde{a}} \text{ for } a \in A\}\ .$$

We often write $\zeta_a = \zeta(a)$. The correspondence $\underline{A} : G \mapsto A(G)$ is called *the superspace associated with* the $\mathbb{Z}_2$-set A. Elements of $A(G)$ is called the *G-points* of $\underline{A}$. This is a covariant functor from the category of superalgebras and homorphisms to that of sets and mappings, i.e., each homorphism $f : G \to G'$ induces a map $A(f)$: $A(G) \to A(G')$ defined by $A(f)(\zeta) = f \circ \zeta$ and satisfies $A(f \circ g) = = A(f) \circ A(g)$ for another homorphism $g : G'' \to G$.

Let U be open subset of $\mathbb{R}^{A_{\bar{0}}}$. For an augmented superalgebra (G, ϵ), we define

$$A^U(G) := \{\zeta \in A(G);\ \epsilon \circ \zeta \in U\}.$$

The correspondence $\underline{A}^U : G \mapsto A^U(G)$ is called the *superdomain of A with the body U*. This is a covariant functor from the category of augmented superalgebras to that of sets. Note that $A(\mathbb{R}) = \mathbb{R}^{A_{\bar{0}}}$ and $A^U(\mathbb{R}) = U$.

2.5. Superpolynomial algebras

Let A be a $\mathbb{Z}_2$-set. Let $\mathbb{R}\{A\}$ be the free associative $\mathbb{R}$-algebra generated by A, i.e., $\mathbb{R}\{A\} := \oplus_{w \in w(A)} \mathbb{R} \cdot w$, with the multiplication defined by

$$(\Sigma\, C'_w w)(\Sigma\, C''_w w) = \Sigma_w (\Sigma_{w'.w''=w}\, C_{w'}\, C_{w''})\, w.$$

Let $\mathbb{R}[A]$ be the quotient of $\mathbb{R}\{A\}$ by the two-sided ideal generated by $\{ab - - (-1)^{\tilde{a}\tilde{b}} ba;\ a, b \in A\}$. Let Z_a denotes the class represented by a. Then we have obviously

PROPOSITION (2.5.1).

(i) $$Z^{w \cdot \pi} = sgn(w, \pi)\, Z^w \text{ for } (w, \pi) \in w_r(A) \times S_r,$$

(ii) $$\mathbb{R}[A] \simeq \oplus_{w \in s\overline{w}(A)} \mathbb{R} \cdot Z^w \simeq \oplus_{\mu \in sm(A)} \mathbb{R} \cdot Z^\mu. \qquad ■$$

Define,

$$\mathbb{R}_r[A] := \oplus_{\mu \in sm_r(A)} \mathbb{R} \cdot Z^\mu, \text{ for } r \in \mathbb{Z}_+,$$

$$\mathbb{R}_r[A] = 0 \text{ for } r < 0.$$

Then $\mathbb{R}[A] = \oplus_r \mathbb{R}_r[A]$ and $\mathbb{R}[A]$ is a $\mathbb{Z}$-graded algebra.

Define

$$\mathbb{R}[[A]] := \Pi_{r \in \mathbb{Z}_+} \mathbb{R}_r[A],$$

whose element is called a *formal power series of the variable Z* and expressed as $f = \Sigma_{\mu \in sm(A)} f_\mu Z^\mu$. The algebra $\mathbb{R}[[A]]$ is a superalgebra by the $\mathbb{Z}_2$-grading

$$\tilde{f} = \kappa \Leftrightarrow f_\mu \neq 0 \text{ only when } \tilde{\mu} = \kappa,$$

and the multiplication $h = f \cdot g$, where

$$h_\kappa = \Sigma_{\kappa = \kappa' + \kappa''} f_{\kappa'} g_{\kappa''}.$$

Let G be an associative algebra and $g : A \to G$ a mapping which satisfies

$$g(a)\, g(b) = (-1)^{\tilde{a}\tilde{b}}\, g(b)\, g(a)$$

for $a, b \in A$. Then an algebra map $ev_g : \mathbb{R}[A] \to G$ is uniquely defined by $ev_g(Z_a) = g_a$ for $a \in A$. For $P \in \mathbb{R}[A]$, we write $P(g) := ev_g(P)$. When g_a, are nilpotent, ev_g can be extended to $\mathbb{R}[[A]]$ in the obvious way.

Example (2.5.2). Let $A = A_0 = \{x\}$ and $G = \mathbb{R}[B]$ with $B = B_1 = \{\theta_1, \theta_2, \theta_3, \theta_4\}$. Let $g \in A(G)$ be defined by

$$g(\theta) = \theta_1 \theta_2 + \theta_3 \theta_4$$

and put

$$P(x) := \exp(x) = \Sigma_{n \in \mathbb{Z}_+} x^n / n! \in \mathbb{R}[[A_0]] = \mathbb{R}[[x]].$$

Then

$$P(\theta_1 \theta_2 + \theta_3 \theta_4) := P(g) = 1 + \theta_1 \theta_2 + \theta_3 \theta_4 + \theta_1 \theta_2 \theta_3 \theta_4. \quad \blacksquare$$

In particular, for a superalgebra G, there is a map

$$P(G) : A(G) \to G$$

for $P \in \mathbb{R}[A]$ defined by $P(G)(\zeta) := ev_\zeta(P)$ for $\zeta \in A(G)$. The correspondence $\underline{P} : G \mapsto P(G)$ is *a natural transformation* i.e., for each homomorphism $f : G \to G'$, We have $f \circ P(G) = P(G') \circ A(f)$.

When $A_{\bar{0}} = \phi$, the algebra $\mathbb{R}[A]$ is called *the Grassmann superalgebra on* $A_{\bar{1}}$, which has a unique augmentation ϵ defined by $\epsilon(Z^\mu) = 0$ for $\mu \neq 0$. Obviously Ker ϵ is nilpotent.

2.6. Smooth superfunctions

For a set I, a subset U of $\mathbb{R}^I$ is called *a primitive* open subset if there is a finite subset J of I and an open subset V of $\mathbb{R}^J$ such that

$$U = \{\zeta = (\zeta_i) \in \mathbb{R}^I;\ \zeta|_J = (\zeta_j; j \in J) \in V\}.$$

A function $f : U \to \mathbb{R}$ is called *smooth* if there is a finite subset J' such that $J \subset J' \subset I$ and $f = g \circ \pi$ for some $g \in C^\infty(U_{J'})$, where

$$U_{J'} := \{\zeta \in \mathbb{R}^{J'};\ \zeta|_J \in V\}$$

and $\pi : U \to U_{J'}$ is a natural projection. The set of all smooth functions on U is denoted by $C^\infty(U)$.

Let A be a $\mathbb{Z}_2$-set. Let U be a primitive open set of $\mathbb{R}^{A_{\bar{0}}}$. Let G be a superalgebra. We define *a smooth superfunction on the superspace* A^U *with coefficients in* G as an element of the superalgebra

$$F_G(A^U) := G \otimes_{\mathbb{R}} C^\infty(U) \otimes_{\mathbb{R}} \mathbb{R}[A_1].$$

When $U = \mathbb{R}^{A_{\bar{0}}}$, we write simply $F_G(A) = F_G(A^U)$.

Remark (2.6.1). One may assume $G = \mathbb{R}$ without the danger of losing the essential points of this paper. ■

We write an element $f \in F_G(A^U)$ as

$$f = \Sigma_{\mu \in sm(A_{\bar{1}})} f_\mu Z^\mu,$$

with $f_\mu \in F_G(A_0^U) = G \otimes C^\infty(U)$. Then f is homogeneous of parity κ if and only if $\tilde{f}_\mu + \tilde{\mu} = \kappa$ for all $\mu \in sm(A_{\bar{1}})$.

A homomorphism $\varphi : G \to G'$ of superalgebras induces

$$F_\varphi := \varphi \otimes 1 : F_G(A^U) \to F_{G'}(A^U).$$

If A' is a subset of A, there is a natural inclusion map

$$\iota : F_G(A') \to F_G(A)$$

defined by

$$\iota(\Sigma_{\mu \in sm(A_1')} f_\mu Z^\mu) = \Sigma_{\mu \in sm(A_1')} i(f_{\bar{\mu}}) Z^\mu,$$

where $i : C^\infty(\mathbb{R}^{A'_{\bar{0}}}) \to C^\infty(\mathbb{R}^{A_{\bar{0}}})$ is the map induced by the projection $\mathbb{R}^{A_{\bar{0}}} \to \mathbb{R}^{A'_{\bar{0}}}$.

2.7. Partial derivatives

For $a \in A$, we define

$$\partial_a = \partial/\partial Z_a : F_G(A^U) \to F_G(A^U)$$

as follows: when $\tilde{a} = \bar{0}$, we put $\partial_a = 1 \otimes \partial_a \otimes 1$, where $\partial_a : C^\infty(U) \to C^\infty(U)$

is the usual differentiation. For $\tilde{a} = \bar{1}$, we define $\partial_a := 1 \otimes 1 \otimes \partial_a$, where $\partial_a : \mathbb{R}[A_{\bar{1}}] \to \mathbb{R}[A_{\bar{1}}]$ is defined by

$$\partial_a(Z_{a(1)} \dots Z_{a(p)}) = \Sigma_{j=1}^{p} (-1)^{j-1} \delta_{aa(j)} Z_{a(1)} \dots \check{Z}_{a(j)} \dots Z_{a(p)},$$

Obviously we have

LEMMA (2.7.1).

(i) $\partial_a \partial_b = (-1)^{\tilde{a}\tilde{b}} \partial_b \partial_a$ *for* $a, b \in A$,

(ii) $\partial^{w:\pi} = sgn(w, \pi)\partial^w$ *for* $(w, \pi) \in w_r(A) \times S_r$. ■

2.8. Taylor expansion of smooth superfunction

Let A be $\mathbb{Z}_2$-set and U a primitive open set of $\mathbb{R}^{A_{\bar{0}}}$. For $f = \Sigma_{\mu'' \in sm(A_{\bar{1}})} f_{\mu''} \otimes Z^{\mu''} \in F_G(A^U)$, define $\hat{f} \in F_G(A^U_{\bar{0}}) \otimes \mathbb{R}[[A]]$ by

$$\hat{f} = \Sigma_{\mu \in sm(A)} (1/\mu'!) \partial^{\mu'} f_{\mu''} \otimes Z^{\mu' + \mu''},$$

$$= \Sigma_{\lambda \in m(A_{\bar{0}})} \Sigma_{\mu \in sm(A_{\bar{1}})} (1/\lambda!) \partial^\lambda f_\mu \otimes Z^\lambda Z^\mu$$

where $\mu' = \mu \mid_{A_{\bar{0}}}$, $\mu'' = \mu \mid_{A_{\bar{1}}}$. Informally, $\hat{f}(y, Z)$ is the Taylor expansion of $f(y + Z_{\bar{0}}, Z_{\bar{1}})$ at $(y, 0)$ where $Z_\kappa = (Z_a; a \in A_{\bar{\kappa}})$. The correspondence $\tau : f \mapsto \hat{f}$ defines a homomorphism of superalgebras $\tau : F_G(A^U) \to F_G(A^U_{\bar{0}}) \otimes \mathbb{R}[[A]]$.

LEMMA (2.8.1). *For* $f \in F_G(A^U)$,

$$(\partial_a \otimes 1)\hat{f} = (1 \otimes \partial_a)\hat{f} = (\partial_a f)\hat{}, \quad a \in A_{\bar{0}},$$

$$(1 \otimes \partial_a)\hat{f} = (\partial_a f)\hat{}, \quad a \in A_{\bar{1}}.$$

Proof. For $a \in A_{\bar{1}}$, we have $(1 \otimes \partial_a) \circ \tau = \tau \circ \partial_a$ obviously. Let $a \in A_{\bar{0}}$. Then $(\partial_a \otimes 1) \circ \tau = \tau \circ \partial_a$ obviously. We have only to check the equality $(\partial_a \otimes 1)\tau = (1 \otimes \partial_a)\tau$:

$$(\partial_a \otimes 1)\hat{f} = \Sigma_\mu (1/\mu'!) \partial_a \partial^{\mu'} f_{\mu''} \otimes Z^\mu$$

$$= \Sigma_{\mu, \mu_a \geq 1} (\mu_a/\mu'!) \partial^{\mu'} f_{\mu''} \otimes Z^{\mu - \delta_a}$$

$$= \Sigma_\mu (1/\mu'!) \partial^{\mu'} f_{\mu''} \otimes \partial_a Z^\mu$$

$$= (1 \otimes \partial_a)\hat{f}. \qquad \text{q.e.d.}$$ ■

Example (2.8.2). Let $A_{\bar{0}} = \{x\}$, $A_{\bar{1}} = \{\theta, \eta\}$ and put

$$f(x, \theta, \eta) = x^2 \theta \eta.$$

Then

$$\hat{f} = x^2 \otimes \theta\eta + 2x \otimes x\,\theta\eta + 1 \otimes x^2\,\theta\eta.$$

We have

$$(\partial_x \otimes 1)\hat{f} = (1 \otimes \partial_x)\hat{f} = (2x \otimes \theta\eta)\hat{} = 2(x \otimes \theta\eta + 1 \otimes \theta\eta),$$
$$(1 \otimes \partial_\theta)\hat{f} = (x \otimes \eta)\hat{} = x^2 \otimes \eta + 2x \otimes x\eta - 1 \otimes x^2\eta. \qquad \blacksquare$$

2.9. Value of superfunctions

Let (G, ϵ) be an admissible augmented superalgebra. Then a G-point ζ of $\underline{A}^U$ defines a homomorphism

$$ev_\zeta : F_G(A^U) \to G$$

as follows: Since $\zeta_0 := \epsilon \circ \zeta\,|_{A_{\bar{0}}} \in U$, there is a homomorphism

$$ev_1 : F_G(A_0^U) := G \otimes C^\infty(U) \to G$$

defined by $ev_1(g \otimes f) = gf(\zeta_0)$. Since $\zeta(a) - \epsilon(\zeta(a))$ is nilpotent for $a \in A$, a homomorphism

$$ev_2 = ev_{\zeta - \epsilon\circ\zeta} : \mathbb{R}[[A]] \to G$$

is defined. Define then $ev_\zeta := \pi \circ (ev_1 \otimes ev_2) \circ \tau$, where $\pi : G \otimes G \to G$ is the multiplication map.

Example (2.9.1). Let A be the $\mathbb{Z}_2$-set of Example (2.8.2) and put

$$f(x, \theta, \eta) = e^x\theta\eta.$$

Let $G = \mathbb{R}[B]$ with $B = B_1 = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ and $\zeta = (\xi_1\xi_2, \xi_3, \xi_4)$ be a G-point of $\underline{A}$. Then the value of f and ζ is given by

$$f(\zeta) = f(\xi_1\xi_2, \xi_3, \xi_4) = (1 + \xi_1\xi_2)\xi_3\xi_4 = \xi_3\xi_4 + \xi_1\xi_2\xi_3\xi_4. \qquad \blacksquare$$

§3. SUPERMAPS

3.1. Supermaps

Let A and I be $\mathbb{Z}_2$-sets. Let U be a primitive open subset of $\mathbb{R}^{A_{\bar{0}}}$. Define

$$F_G(A^U, I) := I(F_G(A^U)),$$

whose elements are called *smooth supermaps with coefficients in G from $\underline{A}^U$ to I.* They are given by $F = (F_i\,; i \in I)$ with $F_i \in F_G(A^U)_{\tilde{i}}$. When $G = \mathbb{R}$, we write simply $F = F_{\mathbb{R}}$.

Suppose (G, ϵ) is an admissible augmented algebra. Then an element F of $F_G(A^U, I)$ induces

$$F : A^U(G) \to I(G)$$

by

$$F(\zeta) := (F_i(\zeta);\ i \in I) \in I(G).$$

When G is augmented by ϵ, the composition $I(\epsilon) \circ F$ define C^∞ map $\bar{F} : U \to \to \mathbb{R}^{I_{\bar{0}}}$. When V is an open subset of $\mathbb{R}^{I_{\bar{0}}}$, we put
$F_G(A^U, I^V) := \{F \in F_G(A^U, I); \bar{F}(U) \subset V\}$.

3.2. Superalgebra homomorphisms induced by supermaps

Suppose (G, ϵ) is an admissible augmented superalgebra. Then an $F \in F_G(A^U, I)$ induces a homomorphism

$$F^* : F_G(I) \to F_G(A^U)$$

defined as follows: Let $\tilde{\epsilon} : F_G(A^U) \to C^\infty(U)$ be the homomorphism defined by

$$\tilde{\epsilon}(\Sigma_{\mu \in sm(A_{\bar{1}})} f_\mu \otimes Z^\mu) = (\epsilon \otimes I) f_0.$$

Then $(\text{Ker}\ \tilde{\epsilon})^N = 0$, for large N. Put

$$\tilde{\epsilon}F := (\tilde{\epsilon}F_i;\ i \in I_{\bar{0}}) \in C^\infty(U)^{I_{\bar{0}}},$$

which defines $(\tilde{\epsilon}F)^* : G \otimes C^\infty(\mathbb{R}^{I_0}) \to G \otimes C^\infty(U)$.

Define $F - \tilde{\epsilon}F := (F_i - \tilde{\epsilon}F_i; i \in I) \in F_G(A^U)$. Since its components are nilpotent, this induces

$$(F - \tilde{\epsilon}F)^* : \mathbb{R}[[I]] \to F_G(A^U).$$

Finally define $F^* := \pi \circ ((\tilde{\epsilon}F)^*) \otimes (F - \tilde{\epsilon}F)^* \circ \tau$, where $\pi : G \otimes C^\infty(U) \otimes F_G(A^U) \to F_G(A^U)$ is the multiplication map and $G \otimes C^\infty(U) = F_G(A^U_{\bar{0}})$ is regarded as a subalgebra of $F_G(A^U)$.

$$\begin{array}{ccc} F_G(I) & \xrightarrow{\tau} & (G \otimes C^\infty(\mathbb{R}^{I_{\bar{0}}})) \otimes \mathbb{R}[[I]] \\ \downarrow F^* & & \downarrow (\tilde{\epsilon}F)^* \otimes (F - \tilde{\epsilon}F)^* \\ F_G(A^U) & \xleftarrow[\pi]{} & (G \otimes C^\infty(U)) \otimes F_G(A^U). \end{array}$$

Note that $F^*Z_i = F_i$ $(i \in I)$. When ζ is a G-point of the superspace A^U, $(F^*f)(\zeta) = f(F(\zeta))$.

Example (3.2.1). Suppose $I = I_0 = \{x\}$, $A = A_1 = \{\theta, \eta\}$, $F_x := \theta\eta$. Then $F^*f = f(0) + f'(0)\theta\eta$, for an element f of $C^\infty(\mathbb{R})$.

PROPOSITION (3.2.2). *For an element F of $F_G(A^U, I)$, the identity*

$$\partial_a \circ F^* = \Sigma_{i \in I}(\partial_a F_i)\, F^* \circ \partial_i$$

holds for every $a \in A$.

For the proof, we need the following

LEMMA (3.2.3). *Suppose the components of $H \in F_G(A^U, I)$ are nilpotent. Let $H^* : \mathbb{R}[[I]] \to F_G(A^U)$ be the induced map. Then*

$$\partial_a \circ H^* = \Sigma_{i \in I}(\partial_a H_i) H^* \circ \partial_i .$$

Proof. Let $w \in \bar{w}_r(I)$. Define $w'_j \in \bar{w}_{j-1}(A)$ and $w''_j \in \bar{w}_{r-j}(A)$ for $j \in N(r)$ by $w = w'_j w(j) w''_j$, and put $w_j := w'_j w''_j \in \bar{w}_{r-1}(A)$. Then

$$\begin{aligned}(\partial_a \circ H^*)Z^w &= \partial_a H^w = \Sigma_{j=1}^r (-1)^{\tilde{a}\tilde{w}'_j} H^{w'_j} \partial_a H_{w(j)} H^{w''_j} \\ &= \Sigma_{j=1}^r (-1)^{w(j)\tilde{w}'_j} (\partial_a H_{w(j)})\, H^{w_j}. \\ (\partial_a H_i)(H^* \circ \partial_i)\, Z^w &= \Sigma_i (\partial_a H_i) H^* (\Sigma_{j=1}^r (-1)^{\widetilde{iw}'_j} \delta_{iw(j)} Z^{w_j}) \\ &= \Sigma_{j=1}^r (-1)^{\tilde{w}'(j) w_j} (\partial_a H_{w(j)})\, H^{w_j}.\end{aligned}$$

q.e.d. ■

Proof of Proposition (3.2.2). Put $F^0 = \tilde{\varepsilon} F$, $F^1 = F - F^0$. For $a \in A_{\bar{0}}$,

and

$$\begin{aligned}\partial_a \circ F^* &= \partial_a \circ \pi \circ F^{0*} \otimes F^{1*} \circ \tau \\ &= \pi \circ (\partial_a \otimes 1 + 1 \otimes \partial_a) \circ (F^{0*} \otimes F^{1*}) \circ \tau \\ &= \pi \circ \{ (\Sigma_{i \in I_0} ((\partial_a F_i^0)\, F^{0*}) \otimes F^{1*}) \quad (\partial_i \otimes 1) \\ &\quad + \Sigma_{i \in I}^{\ell} (F^{0*} \otimes (\partial_a F_i^1)\, F^{1*})(1 \otimes \partial_i) \} \circ \tau \\ &= \Sigma_i (\partial_a F_i^0 + \partial_a F_i^1)\, \pi \circ (F^{0*} \otimes F^{1*}) \circ \tau \circ \partial_i \\ &= \Sigma_i (\partial_a F_i)\, F^* \circ \partial_i .\end{aligned}$$

For $a \in A_1$, the assertion can be proved similarly. q.e.d. ■

3.3. Composition of supermaps

Let A_i $(i = 1, 2, 3)$ be $\mathbb{Z}_2$-sets. Let (G, ϵ) be an admissible augmented superalgebra. Then we can define

$$F_G(A_2, A_3) \times F_G(A_1, A_2) \mapsto F_G(A_1, A_3)$$

by $(F_1, F_2) \mapsto F_1 \circ F_2$, where

$$(F_1 \circ F_2)_a = F_2^*(F_{1a}), \text{ for } a \in A_3.$$

When for an element F_{12} of $F_G(A_1, A_2)$ there exists an $F_{21} \in F_G(A_2, A_1)$ such that $F_{12} \circ F_{21} = id_{A_1}$, $F_{21} \circ F_{12} = id_{A_2}$, the supermap F_{12} is called *a superdiffeomorphism from A_1 to A_2*, and F_{21} is called *the inverse of F_{12}*.

3.4. Lemmas on regular ideals

In this subsection, we introduce the notion of regularity of ideals of the superfunction algebra, when the index set of the coordinates is well-ordered. We prove the existence of the set of canonical generators of such ideals.

First we prepair two lemmas.

Let A be a $\mathbb{Z}_2$-set and B its subset. Define $\iota \in F(B, A)$ by $\iota_a = Z_a$ $(a \in B)$ and $\iota_a = 0$ $(a \in B^c)$. We denote $\iota^* f$ $(f \in F_G(A))$ by $f(Z_a = 0; a \in B^c)$. This is an element of $F_G(B)$ and hence of $F_G(A)$ by our convention $F_G(B) \subset F_G(A)$ (cf. §2.6).

The first lemma is a weak version of the Taylor theorem for the superfunctions.

LEMMA (3.4.1). *For* $f \in F_G(A)$,

$$f = f(Z_a = 0;\ a \in B^c) + \Sigma_{a \in B^c} Z_a f_a$$

where $f_a \in F_G(A)$. *In particular*

$$Ker\ \iota^* = F_G(A) \cdot \{Z_a;\ a \in B^c\}.$$

Proof. Write f as

$$f = \Sigma_{(\lambda,\ \mu) \in sm(B_{\bar{1}}) \times sm(B_{\bar{1}}^c)} f_{\lambda\mu} Z^\lambda Z^\mu .$$

Then f can be written as follows:

$$f = \Sigma_{\lambda \in sm(B_{\bar{1}})} f_{\lambda 0} Z^\lambda + \Sigma_{a \in B_{\bar{1}}^c} Z_a f_a ,$$

where $f_{\lambda\mu} \in G \otimes C^\infty(\mathbb{R}^{A_{\bar{0}}})$, $f_a \in F_G(A)$. By the usual Taylor theorem, we can express $f_{\lambda 0}$ as

$$f_{\lambda 0} = f_{\lambda 0}(Z_a = 0;\ a \in B_{\bar{0}}^c) + \Sigma_{a \in B_{\bar{0}}^c} Z_a f_{\lambda 0 a}$$

with $f_{\lambda 0 a} \in C^\infty(\mathbb{R}^{A_{\bar{0}}})$. Hence

$$f = \Sigma_{\lambda \in sm\,(B_{\bar{1}})} f_{\lambda 0}(Z_a = 0;\ a \in B_0^c)\, Z^\lambda + \Sigma_{a \in B^c} Z_a f_a$$
$$= f(Z_a = 0; a \in B^C) + \Sigma_{a \in B^c} Z_a f_a,$$

where $f_a = \Sigma_{\lambda \in sm\,(B_{\bar{1}})} f_{\lambda 0 a} Z^\lambda$ for $a \in B^c_{\bar{0}}$. q.e.d. ■

The next lemma determines the quotient of the superfunction algebras by the ideals of special form.

LEMMA (3.4.2). *Let $B \subset A$ and $\varphi \in F_G(B^c, B)$. Let $\Phi \in F_G(B^c, A)$ be defined by $\Phi_a = Z_a$ $(a \in B^c)$ and $\Phi_b = \varphi_b$ $(b \in B)$. Then*

(i) *$Ker\ \Phi^*$ is the ideal $\mathscr{I}$ generated by $\{Z_a - \varphi_b;\ b \in B\}$,*

(ii) *Φ^* induces an isomorphism $F_G(A)/\ \mathscr{I} \simeq F_G(B^c)$.*

Proof. Define $\Psi \in F_G(A, A)$ by $\Psi_a = Z_a$ $(a \in B^c)$ and $\Psi_b = Z_b - \varphi_b$ $(b \in$ $\in B)$. Then Ψ is obviously a superdiffeomorphism. Then $\Psi \circ \Phi \in F_G(B^c, A)$ is just the inclusion ι. In fact, for $a \in B^c$

$$(\Psi \circ \Phi)_b = \Phi^* \Psi_b = \Phi^*(Z_b - \varphi_b) = \varphi_b - \varphi_b = 0.$$

Hence

$$Ker\ \Phi^* = Ker(\iota^* \circ (\Phi^{-1})^*)$$
$$= \Psi^*(Ker\ \iota^*)$$
$$= \Psi^*(F_G(A) \cdot \{Z_b;\ b \in B\})$$
$$= F_G(A) \cdot \{Z_b - \varphi_b;\ b \in B\},$$

whence (i). The assertion (ii) is obvious. q.e.d. ■

We call $\Phi^* f$ *the function obtained from f by the substitutions $Z_b = \varphi_b$* $(b \in$ $\in B)$ and denote it by $f(Z_b = \varphi_b; b \in B)$.

Suppose now that the $\mathbb{Z}_2$-set A is given a well-order. For a subset B of A, we define

$$F_G^0(A, B) := \{\varphi \in F_G(A, B);\ \varphi_b \in F_G(A^b) \text{ for } b \in B\},$$

$$F_G^{00}(A, B) := \{\varphi \in F_G^0(A, B);\ \varphi_b \in F_G(A^b \cap B^c) \text{ for } b \in B\}.$$

A subset E of $F_G(A)$ is called *regular (with respect to the order* « $<$ »), if, for some subset B of A,

$$E = E(\varphi) := \{Z_b - \varphi_b;\ b \in B\}$$

with $\varphi \in F^0(A, B)$. The set B is called *the set of principal indices of* E and denoted by $pi(E)$.

Example (3.4.5). Let $A_{\bar 0} = \{x_1, x_2\}, A_{\bar 1} = \{\theta_1, \theta_2, \theta_3\}$ with $x_1 < \theta_1 < \theta_2 < x_2 < \theta_3$. The set $E := \{\theta_3 - x_1\theta_1 x_2, x_2 - x_1\theta_1\theta_2\}$ is regular with $pi(E) = \{\theta_3, x_2\}$ and can be written as $E(\varphi)$ with $\varphi_{\theta_3} := x_1\theta_1 x_2$ and $\varphi_{x_2} := x_1\theta_1\theta_2$. Note that this is not regular with respect to some other orders, e.g., $x_1 < x_2 < \theta_1 < \theta_2 < \theta_3$.

Let B be a subset of A and φ an element of $F_G^0(A, B)$. The ideal generated by $E(\varphi)$ will be denoted by Ideal(φ). The ideals obtained in this way are called *regular with respect to the order* « $<$ », or simply *regular* if it is clear which order is relevant.

Note that the regular ideal $\mathscr{I}$ generated by the regular set E in the Example (3.4.3) can be generated also by the regular set $E(\psi) = \{\theta_3, x_2 - x_1\theta_1\theta_2\}$. This set of generators has the remarkable property that both ψ_{θ_3} and ψ_{x_2} depend neither on θ_3 nor on x_2. The next proposition shows that for every regular ideal we can always select such a canonical set of generators.

PROPOSITION (3.4.4). *Let B be a subset of a well-ordered $\mathbb{Z}_2$-set A and φ an element of $F_G^0(A, B)$. Then*

(i) *there exists unique* $\psi \in F_G^{00}(A, B)$ *such that* Ideal (φ) = Ideal (ψ).

(ii) *If* $\varphi^0 \in F_G^0(A, B)$ *satisfies* $\varphi_b^0 \in$ Ideal(φ) *for* $b \in B$, *then* Ideal(φ^0) = Ideal (φ).

Proof. (i) We prove by induction on b that for all $b' \in B^{(b)}$ there exists $\psi_{b'} \in F_G(A^{b'})$ such that

$$(*) \qquad I_b := F_G(A^{(b)}) \cdot \{Z_{b'} - \varphi_{b'};\ b' \in B^{(b)}\}$$
$$= F_G(A^{(b)}) \cdot \{Z_{b'} - \psi_{b'};\ b' \in B^{(b)}\}$$

For $b = \min B$, it suffices to put $\psi_b = \varphi_b$.

Suppose for some b, the assertion is true for $b' \in B^b$. Let $\Phi \in F_G(A^b \setminus B, A^b)$ be defined by $\Phi_a = Z_a$ if $a \in B$ and $\Phi_{b'} = \psi_{b'}$ for $b' \in A^b \cap B = B^b$. By Lemma (3.4.2),

$$F_G(A^b) \cdot \{Z_{b'} - \psi_{b'};\ b' \in B^b\} = Ker\, \Phi^*,$$

whence the induction hypothesis implies

$$F_G(A^b) \cdot \{Z_{b'} - \varphi_{b'};\ b' \in B^b\} = Ker\, \Phi^*.$$

Let $\psi_b = \Phi^* \varphi_b$. Then $Z_b - \varphi_b \equiv Z_b - \psi_b$ (mod. $\cup_{b' < b} I_{b'}$), whence (*).

To show the uniqueness, define $\Psi \in F_G(A \setminus B, A)$ by $\Psi_a = Z_a$ $(a \in B^c)$, $\Psi_b = \psi_b$ $(b \in B)$. Then

$$Ker\, \Psi^* = F_G(A) \cdot \{Z_b - \psi_b;\ b \in B\}$$
$$= F_G(A) \cdot \{Z_b - \varphi_b;\ b \in B\} = \text{Ideal}\,(\varphi).$$

Suppose $Z_b - f_b \in$ Ideal (φ) for some $f_b \in F_G(A \setminus B)$. Apply Ψ^*, then

$$\psi_b = \Psi^* Z_b = f_b,$$

whence the uniqueness of ψ_b.

(ii) By (i), Ideal (φ^0) = Ideal (ψ^0) for some $\psi^0 \in F_G^{0\,0}(A, B)$. But $Z_b - \psi_b^0 \in$ Ideal (φ) implies $\psi_b^0 = \psi_b$.

Hence Ideal (φ^0) is generated by $Z_b - \psi_b$ $(b \in B)$ and must coincide with Ideal (φ). q.e.d. ■

§4. SYSTEMS OF SUPER DIFFERENTIAL EQUATIONS

We define a system of super differential equations as a set of superfunctions on the «jet superspace». Two systems are called equivalent if they generate the same differential ideal.

Section 4.1. introduces the jet superspace and the jet extension map, Section 4.2. defines the notion of a system of super differential equations and that of its solutions. In Section 4.3 we define the notion of the regularity of the systems of super differential equations, when the sets of the independent variables on the jet space is given an admissible total order.

Hereafter we fix an augmented superalgebra (G, ϵ) and omit the index G from various notations. For example, $F_G(A)$, $F_G(A, B)$ and $F_G^{0\,0}(A, B)$ will be denoted simply by $F(A)$, $F(A, B)$ and $F^{0\,0}(A, B)$ respectively, where B is a subset of a $\mathbb{Z}_2$-set A.

4.1. Jet superspaces

Let A and I be finite $\mathbb{Z}_2$-sets. Put

$$J_r = J_r(A, I) := A \amalg sm_{(r)}(A, I),$$
$$\mathcal{F}_r := F(J_r),$$

for $r \in \mathbb{Z}_+^*$. We write J_∞ simply by J.

Let (G, ϵ) be an admissible augmented superalgebra. For a subset X of $F(J)$, the ideal $F(J) \cdot X$ will be written also by $\mathcal{I}(X)$.

For $W = (w, i) \in w(A) \times I$, we define $u_W \in F(J(A, I))$ as follows:

$$u_W := \begin{cases} 0 & \text{if } w \in sw(A)^c, \\ Z_{(\mu, i)} & \text{if } w = w(\mu) \quad (\mu \in sm(A)), \\ sgn\,(\overline{w}, \pi)\, Z_{(\overline{w}, \pi)}, & \text{if } w = \overline{w} \cdot \pi \ ((\overline{w}, \pi) \in sw_r(A) \times S_r). \end{cases}$$

Obviously u_W is well-defined. Moreover there is a super diffeomorphism

$$U \in F(J(A, I), A \amalg \overline{sw}(A, I))$$

defined by

$$U_a = Z_a, \ U_W = u_W \quad (a \in A, \ W \in \overline{sw}(A, I)).$$

Informally speaking, $(Z_a, u_w \ : \ a \in A, \ W \in \overline{sw}(A, I))$ is another system of coordinates on J.

For any subset $B \subset J(A, I)$, define the jet extension map

$$j^B : F(A, I) \to F(A, B)$$

by

$$(j^B s)_b := \begin{cases} Z_b & \text{if } b \in A \cap B, \\ \partial^{\mu(M)} s_{i(M)} & \text{if } b = M \in sm(A, I), \end{cases}$$

for $s = (s_i ; i \in I) \in F(A, I)$. When $B = J(A, I)$, we write simply $j = j^B$.

Example (4.1.1). Let $A_{\overline{0}} := \{x, y\}$, $A_{\overline{1}} := \{\theta, \eta\}$, $I_{\overline{0}} := \{u\}$, $I_{\overline{1}} := \{v\}$. Then $J(A, I) = A \amalg sm(A, I)$ is described as

$$J(A, I)_{\overline{0}} = \{x, y, u_{x^i y^j \theta^a \eta^b}, v_{x^i y^j \theta^c \eta^d} ;$$

$$0 \leqslant a, b, c, d \leqslant 1, \ a + b \text{ and } c + d + 1 \text{ is even}\},$$

$$J(A, I)_{\bar{1}} = \{\theta, \eta, u_{x^i y^j \theta^a \eta^b}, v_{x^i y^j \theta^c \eta^d};$$
$$0 \leqslant a, b, c, d \leqslant 1, \ a+b \text{ and } c+d+1 \text{ is odd}\}.$$

The jet extention map $j: F(A, I) \to F(A, J(A, I))$ maps the element given by $u = \varphi(x, \theta), v = \psi(x, \theta)$ to that described by

$$u_{x^i y^j \theta^a \eta^b} = \partial_x^i \partial_y^j \partial_\theta^a \partial_\eta^b \varphi, \ v_{x^i y^j \theta^a \eta^b} = \partial_x^i \partial_y^j \partial_\theta^a \partial_\eta^b \psi. \qquad \blacksquare$$

Obviously we have

LEMMA (4.1.2). *For $s \in F(A, I)$ and $(w, i) \in w(A) \times I$, the following identity holds:*

$$(js)^* u_{(w, i)} = \partial^w s_i. \qquad \blacksquare$$

Remark (4.1.3). The space $F(J_\infty(A, I))$ can be identified with the space of all the differential operators from $F(A, I)$ to $F(A)$: In fact each element P of $F(J_\infty(A, I))$ induces a «super differential operator» $D_P = D(P) : F(A, I) \to F(A)$ by the formula $D_P(s) := (js)^*P$.

For example, when $I = N(1 \mid 0)$, $A = A_0 = \{x, y\}$, we have

$$D(u_x) = \partial/\partial x, \qquad D(u_x^2)(s) = (\partial s/\partial x)^2,$$
$$D(u_{x^2}) = \partial^2/\partial x^2, \ D(u_{xy}) = \partial^2/\partial x \partial y.$$

Note that D_P is linear if and only if P is linear with respect to the variables in $sm(A, I)$. ■

4.2. Systems of super differential equations

Hereafter we fix $\mathbb{Z}_2$-sets A and I. We put $\mathscr{F} = F(J)$.

By the Remark (4.1.3), the concept of the general systems of super differential equations can be formulated as follows.

A subset $\mathscr{E} \subset F_G(J(A, I))$ is called *a system of superdifferential equations with coefficients in G on the supermaps from A to I.* A supermap $s \in F(A, I)$ is called *a solution of $\mathscr{E}$* if

$$(js)^*E = 0, \ \text{for } E \in \mathscr{E}.$$

In other words s is a solution of $\mathscr{E}$ if and only if s satisfies all of the super differential equations:

$$D(P)s = 0, \ \text{for } P \in \mathscr{E}.$$

The set of all the solutions of $\mathscr{E}$ is denoted by $Sol(\mathscr{E})$.

Example (4.2.1). Suppose $G = \mathbb{R}$, $A = N(0 \mid 4)$

$$= \{\theta_1, \theta_2, \theta_3, \theta_4\},\ I = N(1 \mid 0) = \{u\}.$$

Then

$$\mathscr{E} = \{u_{\theta_1 \theta_2} + u_{\theta_3 \theta_4}\}$$

defines the equation:

$$\partial^2 u/\partial\theta_1 \partial\theta_2 + \partial^2 u/\partial\theta_3 \partial\theta_4 = 0.$$

Since this is a linear equation, the solution space is an $\mathbb{R}$-linear subspace of $\mathbb{R}[\theta_1, \ldots, \theta_4]$ and is spanned by $1, \theta_1, \theta_2, \theta_3, \theta_4, \theta_1\theta_3, \theta_1\theta_4, \theta_2\theta_3, \theta_2\theta_4$, $\theta_1\theta_2 - \theta_3\theta_4$ as is easily seen. ∎

LEMMA (4.2.2). *Let* $\mathscr{E}' := \{E_\kappa; E \in \mathscr{E}, \kappa \in \mathbb{Z}_2\}$. *Then*

$$Sol(\mathscr{E}) = Sol(\mathscr{E}').$$

Proof. Obvious since $((js)^*E)_\kappa = (js)^*E_\kappa$ $(\kappa \in \mathbb{Z}_2)$. q.e.d. ∎

Hereafter we assume a system of super differential equations $\mathscr{E}$ consists of homogeneous elements.

Two systems $\mathscr{E}_1, \mathscr{E}_2 \subset F$ are called *equivalent* if the ideals generated by them coincides. We denote then $\mathscr{E}_1 \sim \mathscr{E}_2$. Obviously equivalent systems have the same solution space.

4.3. Regular systems

Fix now a linear order on A and an admissible one on $m(A, I)$. Define a linear order « $<$ » on J extending those on A and $sm(A, I)$ by $a < M$ for $a \in A$ and $M \in sm(A, I)$. Unless otherwise specified, we assume that the set $sm(A, I)$ is given the order (i) of Example (1.3.3) when A and I are given linear orders.

Let Ξ be s subset of $sm(A, I)$. Recall that Ξ is a $\mathbb{Z}_2$-set by the restricted $\mathbb{Z}_2$-grading. A system is called *regular* if it is equivalent to $\mathscr{E}(\varphi)$ for some Ξ and $\varphi \in F^0(J, \Xi)$. The set Ξ is called *the set of principal indices of* $\mathscr{E}$, and is denoted by $pi(\mathscr{E})$. Note that the notion of regularity depends heavily on the choice of orders on $sm(A, I)$.

Example (4.3.1). Let $A_{\bar{0}} = \{x, y\}$, $A_{\bar{1}} = \{\theta, \eta\}$, $I = I_{\bar{0}} = \{u\}$. Let

$$\mathscr{E} = \{\theta u_x + u_\theta + f,\ \eta u_y + u_\eta + g\},$$

where f and g are elements of $C^\infty(A)$. We give the order $x < y < \theta < \eta$. If we put $\Xi = \{\theta, \eta\}$ and define an element φ of $F^0(J, \Xi)$ by $\varphi_\theta = -\theta u_x - f$ and $\varphi_\eta = -\eta u_y - g$, then $\mathscr{E} = \mathscr{E}(\varphi)$, and the system $\mathscr{E}$ is regular. However note that if we adopt the order: $x > y > \theta > \eta$, then this system is not regular. ■

By Lemma (3.4.2), we have

LEMMA (4.3.2). *Let $\mathscr{E}$ be a regular system of super differential equations with $\Xi = pi(\mathscr{E})$. Then there exists a unique $\psi \in F^{00}(J, \Xi)$ such that $\mathscr{E} \sim \mathscr{E}(\psi)$.*

Furthermore, if $\varphi \in F^0(J, \Xi)$ satisfies $\mathscr{E}(\varphi) \subset \mathscr{F} \cdot \mathscr{E}$, then $\mathscr{E}(\varphi) \sim \mathscr{E}$. ■

REMARK. The assumption of regularity is not restrictive: If $\mathscr{E} = \{\varphi_1, \dots, \varphi_\ell\}$ and «the symbol submodule at $\zeta \in A(G)$» is regular of rank ℓ, then around ζ, the system $\mathscr{E}$ is regular. This follows from the implicit function theorem for the supermaps. ■

§5. FORMAL GROEBNER INTEGRABILITY

As in the previous section, we fix finite $\mathbb{Z}_2$-sets A and I. We put $J_r = J_r(A, I)$ and $\mathscr{F}_r = F(J_r)$, for $r \in \mathbb{Z}_+^*$. Fix a linear order on A and an admissible one on $m(A, I)$, and extend it to J_∞ as in Section 4.3. For $M \in m(A, I)$, we put $\mathscr{F}^M := F(J^M_\infty)$, $\mathscr{F}^{(M)} := F(J^{(M)}_\infty)$. Note that $J^{(M)}_\infty = J^M_\infty$ when M does not belong to $sm(A, I)$.

5.1. Extension of partial derivatives

For $a \in A$, we define $d_a : \mathscr{F}_\infty = F(J_\infty(A, I)) \to \mathscr{F}_\infty$ by

$$d_a F = \partial_a F + \Sigma_{(\mu,i)\in sm(A,I)}\, u_{(aw(\mu),i)}\, \partial_{(\mu,i)} F,$$

for $F \in \mathscr{F}_\infty$.

LEMMA (5.1.1). *Let $s \in F(A, I)$. Then*

$$(js)^* d_a F = \partial_a (js)^* F$$

for $a \in A$ and $F \in \mathscr{F}_\infty$.

Proof. By Propositions (3.2.2) and (4.1.2),

$$(js)^* d_a F = (js)^*(\partial_a F + \Sigma_{(\mu,i)}\, u_{(aw(\mu),i)}\, \partial_{(\mu,i)} F)$$

$$= \Sigma_{b \in A} \partial_a Z_b (js)^* \partial_b F + \Sigma_{(\mu,i)} \partial_a (\partial^\mu s_i)(js)^* \partial_{(\mu,i)} F$$

$$= \Sigma_{M \in J(A,I)} \partial_a ((js)_M)(js)^*(\partial_M F)$$

$$= \partial_a (js)^* F. \qquad \text{q.e.d.}$$

■

LEMMA (5.1.2). *Suppose* $(\mu, M) \in sm(A) \times sm(A, I)$ *satisfies* $\mu + M \in sm(A, I)$. *Then*

$$d^\mu (\mathscr{F}^M_\infty) \subset \mathscr{F}_\infty^{\mu + M}.$$

Proof. We may assume $\mu = \delta_a$ $(a \in A)$. Since $M' < M$ implies

$$\delta_a + M' < \delta_a + M,$$

the condition $F \in \mathscr{F}^M$ implies $d_a F \in \mathscr{F}_\infty^{\delta_a + M}$ obviously.

q.e.d. ■

5.2. Formal Gröbner integrability

Define for $E \subset \mathscr{F}_\infty$ and $n \in \mathbb{Z}^*_+$

$$\rho_n(\mathscr{E}) := \{d^\mu E; \mu \in sm_{(n)}(A),\ E \in \mathscr{E}\}$$

By Lemma (5.1.1), $Sol(\mathscr{E}) = Sol(\rho_n(\mathscr{E}))$ for $n \in \mathbb{Z}^*_+$.

DEFINITION (5.2.1). A regular system $\mathscr{E}$ is called *formally Gröbner integrable* if $\rho_\infty(\mathscr{E})$ is a regular system with

$$pi(\rho_\infty(\mathscr{E})) = \rho_\infty(pi(\mathscr{E})).$$

REMARK (5.2.2). We have always

$$pi(\rho_\infty(\mathscr{E})) \supset \rho_\infty(pi(\mathscr{E}))$$

by Lemma (5.1.2). However the other inclusion is not necessarily true since the ideal generated by $\rho_\infty(\mathscr{E})$ may contain nontrivial compatibility conditions as is illustrated by the following simple example. ■

Example (5.2.3). Let $A_{\bar{0}} = \{x\}$, $A_{\bar{1}} = \{\theta\}$, $I = I_{\bar{0}} = \{u\}$, $\mathscr{E} = \{f: \ = \theta u_x + u_\theta\}$. Then $\mathscr{E}$ is regular with $pi(\mathscr{E}) = \{\theta\}$, and hence $\rho_\infty pi(\mathscr{E}) = \{x^n \theta; n \in \mathbb{Z}_+\}$. But $d_\theta f = u_x$ implies that $x \in pi(\rho_\infty(\mathscr{E}))$, whence $pi(\rho_\infty(\mathscr{E}))$ coinsides with $sm(A)$ and properly contains $\rho_\infty pi(\mathscr{E})$. In particular $\mathscr{E}$ is not formally Gröbner integrable. ■

The main theme of this paper is to give various practical sufficient conditions for the formal Gröbner integrability of regular systems.

In the rest of this section we give simple rephrases of the formal Gröbner integrability.

LEMMA (5.2.4). *Let* $\mathscr{E} = \mathscr{E}(\varphi)$ *with* $\varphi \in F^0(J_\infty, \Xi)$. *Put* $\tilde{\Xi} := \rho_\infty(\Xi)$. *Choose, for each* $\overline{M} \in \tilde{\Xi}$, *an element* (μ, M) *of* $sm(A) \times \Xi$ *such that* $\mu + M = \overline{M}$ *and put* $\tilde{\varphi}_{\overline{M}} := d^M \varphi_M \in \mathscr{F}^{\overline{M}}_\infty$. *Then* $\tilde{\varphi} \in F^0(J_\infty, \tilde{\Xi})$ *and* $\mathscr{E}$ *is formally Gröbner integrable if and only if* $\rho_\infty(\mathscr{E})$ *is equivalent to its subset* $\mathscr{E}(\tilde{\varphi})$.

Proof. If $\rho_\infty(\mathscr{E}) \sim \mathscr{E}(\tilde{\varphi})$, then $\mathscr{E}$ is compatible by definition.
The converse follows from the following lemma. q.e.d. ∎

LEMMA (5.2.5). *Suppose* $\mathscr{E}$ *is formally Gröbner integrable. Let* $\varphi \in F^0(J_\infty, \tilde{\Xi})$ *satisfies* $\varphi_M \in \mathscr{I}(\rho_\infty(\mathscr{E}))$ *for* $M \in \tilde{\Xi}$. *Then* $\mathscr{E}(\varphi)$ *is equivalent to* $\rho_\infty(\mathscr{E})$.

Proof. Let $\rho_\infty(\mathscr{E}) \sim \mathscr{E}(\psi)$ with $\psi \in F^0(J, \tilde{\Xi})$. Then $\mathscr{E}(\varphi) \subset \mathscr{I}(\psi)$, whence by (ii) of Proposition (3.4.3), we have $\mathscr{I}(\varphi) = \mathscr{I}(\psi)$. q.e.d. ∎

LEMMA (5.2.6). *For* $M \in m(A, I)$ *and* $\varphi \in F^0(J, \tilde{\Xi})$, *the ideal of* $\mathscr{F}^{(M)}_\infty$:

$$\mathrm{Ideal}^{(M)}(\varphi) := \mathscr{I}(\varphi) \cap \mathscr{F}^{(M)}_\infty$$

is generated by $\{Z_N - \varphi_N; N \in \tilde{\Xi} \cap J^{(M)}_\infty\}$.

Proof. We may assume $\varphi \in F^{00}(J_\infty, \tilde{\Xi})$. Define $\Phi \in F(J_\infty, J_\infty)$ by $\Phi_N := Z_N$ $(N \in \tilde{\Xi}^c)$, $Z_N - \varphi_N$ $(N \in \tilde{\Xi})$. The induced homomorphism Φ^* is an automorphism satisfying $\Phi^* \mathscr{F}^{(M)}_\infty = \mathscr{F}^{(M)}_\infty$, $\Phi^* Z_N = Z_N - \varphi_N$ $(N \in \tilde{\Xi})$. Hence we may assume $\varphi_N = 0$ for $N \in \tilde{\Xi}$. Let

$$f = \Sigma_{N \in \tilde{\Xi}} f_N Z_N \in \mathscr{F}^{(M)}_\infty.$$

Since $f = f(Z_L = 0,\ L > M)$, we have

$$f = \Sigma_{N \leqslant M} f_N(Z_L = 0, L > M) Z_N$$

which belongs to the ideal of $\mathscr{F}^{(M)}_\infty$ generated by

$$\{Z_N; N \in \tilde{\Xi} \cap J^{(M)}_\infty\}.$$ q.e.d. ∎

COROLLARY (5.2.7). *Let* $\varphi \in F^0(J_\infty, \Xi)$ with $\Xi \subset sm(A, I)$. *If* $f \in \mathscr{F}^{(M)}_\infty$ *belongs to* $\mathscr{I}(\varphi)$, *then* f *belongs to the ideal generated by*

$$\{Z_N - \varphi_N; N \in \tilde{\Xi} \cap J_\infty^{(M)}\}.$$ ∎

§6. CRITERIONS FOR THE FORMAL GROEBNER INTEGRABILITY

6.1. sm-subsets of sm(A, I)

Let A and I be $\mathbb{Z}_2$-sets. Fix an admissible order « $<$ » on $sm(A, I)$.

For $M \in sm(A, I)$ and $\Xi \subset sm(A, I)$, we define

$$\Xi - M := \{\mu \in sm(A);\ \mu + M \in \Xi\}.$$

When Ξ is an sm-subset, $\Xi - M$ is an sm-subset of $sm(A)$.

Let Ξ be a finite subset of $sm(A, I)$. Let Ξ be given a linear order « $<$ ». We define *the set of multiplicative indices*

$$ml\ t(M) = ml\ t_\Xi(M) \subset sm(A)$$

of $M \in \Xi$ *with respect to the order* « $<$ » as follows:

$$ml\ t(M) := [\{\rho_\infty\{M' \in \Xi; M' < M\} - M\} \cup \{sm(A, I)^c - M\}]^c.$$

The obviously the complement $ml\ t(M)^c$ is an sm-subset.

Define

$$ml\ t(A, \Xi) := \{(\mu, M) \in sm(A, \Xi);\ \mu \in ml\ t(M)\}.$$

By the definition of $ml\ t(A, \Xi)$, we have

LEMMA (6.1.1).

(i) *The map* $(\mu, M) \mapsto \mu + M$ *induces a bijection* ν *from* $ml\ t(A, \Xi)$ *to* $\rho_\infty(\Xi)$.

(ii) *For* $(\mu, M) \in \{ml\ t(A, \Xi)\}^c$ *with* $\mu + M \in \rho_\infty(\Xi)$, *let* $\nu^{-1}(\mu + M) = (\mu', M')$. *Then* $M' < M$. ∎

Example (6.1.2). Let $A_{\bar{0}} = \{x, y\}$, $A_{\bar{1}} = \{\theta, \eta\}$, $I = N(1 \mid 0)$ and identify $sm(A, I)$ with $sm(A)$. Let

$$\Xi := \{x^3, xy, x^2\theta, \eta\}$$

and give the order:

$$x^3 > xy > x^2\theta > \eta.$$

Then

$$\{ml\ t(\eta)\}^c = \rho_\infty\{\eta\},$$
$$\{ml\ t(x^2\theta)\}^c \doteq \rho_\infty\{\eta, \theta\},$$
$$\{ml\ t(xy)\}^c = \rho_\infty\{\eta, x\theta\},$$
$$\{ml\ t(x^3)\}^c = \rho_\infty\{\eta, \theta, y\},$$

and hence

$$ml\ t(\eta) = sm\{x, y, \theta\},$$
$$ml\ t(x^2\theta) = sm\{x, y\},$$
$$ml\ t(xy) = \{y^a\theta, x^a y^b; a, b \in \mathbb{Z}_+\},$$
$$ml\ t(x^3) = sm\{x\}.$$

We have the following disjoint union decomposition:

$$\rho_\infty \Xi = \eta \cdot ml\ t(\eta) \amalg x^2\theta \cdot ml\ t(x^2\theta) \amalg xt \cdot ml\ t(xy) \amalg x^3 \cdot ml\ t(x^3).$$

Here we denoted multiplicatively the set generated by subsets S_1 and S_2 of $sm(A)$ by $S_1 \cdot S_2$. ■

Let $\Gamma \subset sm(A)$ be a subset. An element γ of Γ is called *primitive* if γ does not belong to $\rho_\infty(\Gamma \setminus \{\gamma\})$. Let $prim(\Gamma)$ denote the set of all the primitive elements of Γ. As is easily observed, the subset of $sm(A)$ generated by a subset Γ is also generated by $prim(\Gamma)$.

For an sm-subset Ξ of $sm(A)$, the set $prim(\Xi)$ will be denoted by *Basis* (Ξ).

Example (6.1.3). Let A be as in Example (6.1.2). Then

$$prim(\{x^2, xy^4, y^3, \theta\eta, x^3\eta\}) = \{x^2, y^3, \theta\eta\}.$$ ■

PROPOSITION (6.1.4). *The set prim*(Γ) *is a finite.*

Proof. For every pair $\gamma, \gamma' \in prim(\Gamma)$, neither $\gamma \ll \gamma'$ nor $\gamma' \ll \gamma$ is true. Hence by Theorem of Riquier (cf. p. 147 of [Ritt]) must be a finite set. q.e.d. ■

Example (6.1.5). In the Example (6.1.2), we have

$$Basis(\{ml\ t(\eta)\}^c) = \{\eta\},$$

$$Basis(\{ml\ t(x^2\theta)\}^c) = \{\eta, \theta\},$$

$$Basis(\{ml\ t(xy)\}^c) = \{\eta, x\theta\},$$

$$Basis(\{ml\ t(x^3)\}^c) = \{\eta, \theta, y\}. \qquad \blacksquare$$

A subset Ξ of $sm(A, I)$ is called *1-acyclic* if $Basis(\{ml\ t(A, \Xi)\}^c) \subset sm_1(A,\Xi)$. The following is obvious from the definitions:

REMARK. The term «acyclic» comes from the fact that if Ξ is 1-acyclic, then the «super Koszul cohomology» of the sm-subset generated by $\{Z^\alpha;\ \alpha \in \Xi\}$ is trivial. $\blacksquare$

PROPOSITION (6.1.6).

(i) *A subset Ξ of $sm(A, I)$ is 1-acyclic if and only if for every element M of it, there exists a subset A_M of A such that $ml\ t(M) = sm(A_M)$.*

(ii) *If Ξ is 1-acyclic, then*

$$\rho_\infty(\Xi) = \amalg_{M\in\Xi}(sm(A_M) + M). \qquad \blacksquare$$

Example (6.1.7). Let A and I be as in the Example (6.1.2). Let

$$\Xi_1 := \{x^3, xy, x\theta, \eta\}$$

with the order $x^3 > xy > x\theta > \eta$. Then this is 1-acyclic. In fact

$$Basis(\{ml\ t(\eta)\}^c) = \{\eta\},$$

$$Basis(\{ml\ t(x\theta)\}^c) = \{\theta, \eta\},$$

$$Basis(\{ml\ t(xy)\}^c) = \{\theta, \eta\},$$

$$Basis(\{ml\ t(x^3)\}^c) = \{\theta, \eta, y\}.$$

The multiplicative sets are described as follows:

$$ml\ t(\eta) = sm\{x, y, \theta\},$$

$$ml\ t(x\theta) = ml\ t(xy) = sm\{x, y\},$$

$$ml\ t(x^3) = sm\{x\}. \qquad \blacksquare$$

6.2. Precompatibility

Let Ξ be a finite subset of $sm(A, I)$ and $\mathscr{E} \sim \mathscr{E}(\varphi)$ for some $\varphi \in F^0(J_\infty, \Xi)$.

We give Ξ a linear order which may not be the restriction of that of $sm(A, I)$. Let $ml\ t(M)$ be defined with respect to this order.

We denote $\rho_\infty(\Xi)$ by $\tilde{\Xi}$. For each element $\bar{M}$ of $\tilde{\Xi}$, put

$$\tilde{\varphi}_{\bar{M}} := d^\mu \varphi_M ,$$

$$\psi_{\bar{M}} := Z_{\bar{M}} - \tilde{\varphi}_{\bar{M}} ,$$

where $(\mu, M) = \nu^{-1}\bar{M} \in ml\ t(A, \Xi)$ (cf. Lemma (6.1.1)).

DEFINITION (6.2.1). A system of super differential equation $\mathscr{E}$ is called *precompatible* if

$$d^\mu \psi_M \in \mathscr{I}(\tilde{\varphi})$$

for all $(\mu, M) \in Basis\ (\{ml\ t(A, \Xi)\}^c)$, where

$$\tilde{\varphi} = (\tilde{\varphi}_M ; \bar{M} \in \tilde{\Xi}) \in F^0(J_\infty, \tilde{\Xi}). \qquad \blacksquare$$

Note that this is a finite number of conditions by Proposition (6.1.4). The following example gives an example of systems of super differential equations which are not precompatible.

Example (6.2.2). Let $A_{\bar{0}} = \{x, y\}$, $A_{\bar{1}} = \{\theta\}$, $I = I_{\bar{0}} = \{u\}$. Introduce the order $\theta > x > y$ and give $sm(A, I) = sm(A)$ the standard order (cf. beginning of Section 4.3). Let $\Xi = \{\theta, y\}$ and $\varphi \in F^0(J_\infty, \Xi)$ be defined as

$$\varphi_\theta = \theta u_x + f(x, y, \theta, u),$$

$$\varphi_y = g(x, y, \theta, u),$$

with $f, g \in C^\infty(A \amalg I)$. The system $\mathscr{E}(\varphi)$ is the following system of equations

$$\partial u/\partial\theta + \theta\,\partial u/\partial x = f(x, y, \theta, u),$$

$$\partial u/\partial y = g(x, y, \theta, u).$$

Define in Ξ the order $\theta' > y$. Then

$$\{ml\ t(\theta)\}^c = \rho_\infty\{\theta\},$$

$$\{ml\ t(y)\}^c = \rho_\infty\{\theta\},$$

$$\rho_\infty(\Xi) = \rho_\infty\{\theta, \eta\} = \{\theta x^a y^b,\ x^a y^{b+1};\ a, b \geqslant 0\}.$$

Hence

$$Basis(\{ml\ t(A, \Xi)\}^c) = \{(\theta, \theta), (\theta, y)\} \subset sm(A) \times \Xi.$$

On the other hand,

$$ml\ t(\theta) = ml\ t(y) = sm\{x, y\},$$
$$\tilde{\Xi} = \{\theta x^a y^b,\ x^a y^{b+1};\ a, b \in \mathbb{Z}_+\},$$

and

$$\psi_{\theta x^a y^b} = u_{\theta x^a y^b} + d_x^a d_y^b(\theta u_x - f),$$
$$\psi_{x^a y^{b+1}} = u_{x^a y^{b+1}} - d_x^a d_y^b g.$$

Thus $\mathscr{E}(\varphi)$ is precompatible if and only if $d_\theta(u_\theta + \theta u_x - f)$ and $d_\theta(u_y - g)$ belongs to the ideal $\mathscr{I}(\tilde{\varphi})$. However, we have modulo $\mathscr{I}(\tilde{\varphi})$

$$d_\theta(u_\theta + \theta u_x - f) \equiv u_x - (\theta f_x + f_\theta - ff_u),$$
$$d_\theta(u_y - g) \equiv f_y - gf_u - \theta g_x - g_\theta - fg_u,$$

whence the equation $\mathscr{E}(\varphi)$ is not precompatible. ■

For $N \in ml\ t(A, \Gamma)$, let $\mathrm{Ideal}^{(N)}(\tilde{\varphi})$ be the ideal of $\mathscr{F}^{(N)}$ generated by $\{\psi_{\bar{M}};\ \bar{M} \leqslant N\}$. Since $d^\mu \psi_M \in \mathscr{F}^{(\mu + M)}$, Corollary (5.2.7) implies the following lemma.

LEMMA (6.2.2). *A system $\mathscr{E}$ is precompatible if and only if*

$$d^\mu \psi_M \in \mathrm{Ideal}^{(\mu + M)}(\tilde{\varphi})$$

for all $(\mu, M) \in Basis(\{ml\ t(A, \Xi)\}^c)$. ■

6.3. Formal Cartan-Kähler Theorem

Let Ξ be a finite subset of $sm(A, \Gamma)$ and $\mathscr{E} = \mathscr{E}(\varphi)$ for some $\varphi \in F^0(J_\infty, \Xi)$.

THEOREM (6.3.1). *The system $\mathscr{E}$ is formally Gröbner integrable if and only if it is precompatible.*

Proof. We use the notations of the previous section. Suppose $\mathscr{E}$ is formally Gröbner integrable. Then $\rho(\mathscr{E}) \sim \mathscr{E}(\tilde{\varphi})$, where $\tilde{\varphi} \in F^0(J_\infty, \tilde{\Xi})$ is the element defined in the previous section. Then

$$d^\mu \psi_M \in \mathscr{F}^{(\mu + M)} \cap \mathscr{I}(\tilde{\varphi}) = \mathrm{Ideal}^{(\mu + M)}(\tilde{\varphi}).$$

by Lemma (5.1.2).

Conversely suppose $\mathscr{E}(\varphi)$ is precompatible. We prove by induction on $(\mu, M) \in$ $\in sm(A, \Xi)$, that

$$(1)_{\mu, M} \qquad d^\mu \psi_M \in \mathrm{Ideal}^{(\mu + M)}(\tilde{\varphi}).$$

Here the order of $sm(A, \Xi)$ is the pull-back of the lexicographical ordering of $sm(A, \Xi) \times \Xi$ by the map $(\mu, M) \mapsto (\mu + M, M)$. This order is obviously admissible.

For the minimal element $(0, M_0)$ $(M_0 := \min \Xi)$ and for $(\mu, M) \in ml\ t(A, \Xi)$, the assertion $(1)_{\mu,M}$ is trivial. For $(\mu, M) \in Basis(\{ml\ t(A, \Xi)\}^c)$, the assertion $(1)_{\mu,M}$ is valid by the precompatibility of $\mathscr{E}$.

Let $(\bar{\mu}, \bar{M}) \in ml\ t(A, \Xi) \cup Basis(\{ml\ t(A, \Xi)\}^c)$.

Suppose $(1)_{\mu,M}$ is valid for $(\mu, M) < (\bar{\mu}, \bar{M})$. There exist $\lambda, \mu \in sm(A)$ such that $(\mu, \bar{M}) \in Basis(\{ml\ t(A, \Xi)\}^c)$ and $\lambda + \mu = \bar{\mu}$. Then

$$d^{\mu} \psi_{\bar{M}} \in \mathrm{Ideal}^{(\mu + \bar{M})}(\tilde{\varphi}).$$

It suffices to show

$$d^{\lambda}\, \mathrm{Ideal}^{(\mu + \bar{M})}(\tilde{\varphi}) \subset \mathrm{Ideal}^{(\bar{\mu} + \bar{M})}(\tilde{\varphi}),$$

for which we have only to show

$$d^{\lambda} \psi_N \in \mathrm{Ideal}^{(\bar{\mu} + \bar{M})}(\tilde{\varphi}),$$

for $N \in \Xi^{(\mu + \bar{M})}$. Let $\nu^{-1}(N) = (\mu', M') \in ml\ t(A, \Xi)$. Then

$$\psi_N = d^{\mu'} \psi_{M'}.$$

We have $(\mu', M') < (\mu, \bar{M})$.

In fact (i) if $N < \mu + \bar{M}$, then $\mu' + M' = N < \mu + \bar{M}$, *whence* $(\mu', M') < (\mu, \bar{M})$. (ii) Suppose $N = \mu + \bar{M}$. Since $M' < \bar{M}$ and $\mu' + M' = N = \mu + \bar{M}$, we have $(\mu' M') < (\mu, \bar{M})$. Hence $(\lambda + \mu', M') < (\lambda + \mu, \bar{M}) = (\bar{\mu}, M)$ in any case. Therefore

$$d^{\lambda} \psi_N = d^{\lambda + \mu'} \psi_{M'} \in \mathrm{Ideal}^{(\lambda + \mu' + M')}(\tilde{\varphi}) \subset \mathrm{Ideal}^{(\bar{\mu}, \bar{M})}(\tilde{\varphi})$$

by the induction hypothesis. q.e.d. ■

6.4. Involutive systems of super differential equations

Suppose that a regular system is homogeneous, i.e., its principal index set Ξ of a regular system lies in $sm_r(A, I)$ for some $r \in \mathbb{Z}_+$ and that Ξ is 1-acyclic, i.e., the nonmultiplicative set $ml\ t(\Xi)^c$ is generated by elements of degree 1. Then the precompatibility of $\mathscr{E}$ can be rephrased in a different way, close to the classical involutivity.

DEFINITION (6.4.1). A regular system $\mathscr{E}$ is called *involutive* if the following three conditions hold.

(i) $\Xi := pi(\mathscr{E}) \subset sm_r(A, \Gamma)$,

(ii) Ξ is 1-acyclic, i.e., $Basis(ml\ t(\Xi)^c) \subset sm_1(A, \Gamma)$,

(iii) $\mathscr{E}$ is precompatible.

Suppose $\Xi \subset sm_r(A, \Gamma)$ and Ξ is 1-acyclic. Let $\mathscr{E} = \mathscr{E}(\varphi)$ with $\varphi \in F(J_\infty, \Xi)$. Define $\tilde{\varphi} \in F^0(J_\infty, \rho_\infty(\Xi))$ and $\psi \in F(J_\infty, \rho_\infty(\Xi))$ as in §6.2. Put

$$\bar{\rho}_1(\mathscr{E}) := \{\psi_N; N \in \rho_1(\Xi)\}.$$

Then under the conditions (i) and (ii), the precompatibility can be rephrazed as follows:

PROPOSITION (6.4.2). *Suppose $\Xi \subset sm_r(A, \Gamma)$ is 1-acyclic and let $\varphi \in F^0(J_\infty, \Xi)$. The regular system $\mathscr{E} = \mathscr{E}(\varphi)$ is involutive if and only if $\bar{\rho}_1(\mathscr{E})$ and $\rho_1(\mathscr{E})$ generate the same ideal of $\mathscr{F}_{r+1}$.*

Proof. By the 1-acyclicity of Ξ,

$$\{ml\ t(A, \Xi)\}^c \cap sm_1(A, \Xi) = Basis(\{ml\ t(A, \Xi)\}^c).$$

Suppose $\mathscr{E}$ is involutive. Then for $(a, M) \subset sm_1(A, \Xi)$,

$$d_a \psi_M \in \text{Ideal}(\tilde{\varphi}).$$

Then, by Corollary (5.2.7), $d_a \psi_M$ is in the ideal generated by $\bar{\rho}_1(\mathscr{E})$.

Conversely suppose $d_a \psi_M \in \mathscr{F}_1 \cdot \bar{\rho}_1(\mathscr{E})$ for $(a, M) \in sm_1(A, \Xi)$. Then $d_a \psi_M \in$ $\in \text{Ideal}(\tilde{\varphi})$ for $a \in Basis(\{ml\ t(M)\}^c)$, whence $\mathscr{E}$ is precompatible. q.e.d. ■

6.5. Good involutive systems of super differential equations

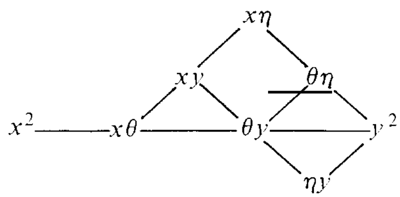
Since the 1-acyclicity of subsets of $sm_r(A, \Gamma)$ is rather difficult to check, we give a practical sufficient condition for a subset Ξ of $sm_r(A, \Gamma)$ to be 1-acyclic. We note that when $A_{\bar{1}}$ is void, this condition holds for «generic» regular systems of differential equations.

Let « $<$ » be a linear order on A. This induces the following partial order, denoted again by « $<$ », on $sm(A, \Gamma)$:

$$M < M' \Leftrightarrow M' = \delta_a - \delta_b + M \text{ for some } a, b \in A \text{ with } b < a.$$

Example. Let $A_{\bar{0}} = \{x, y\}$, $A_{\bar{1}} = \theta, \eta$ with $x > \theta > y > \eta$.

Then on $sm_2(A, \Gamma)$, the induced partial order can be depicted as follows:



where smaller elements lie right to larger ones. ∎

A subset Ξ of $sm_r(A, I)$ is called *good* if there is a linear order « $<$ » on A such that $M \in \Xi$ implies $M' \in \Xi$ for all $M' \in sm_r(A, I)$ with $M' > M$.

For $\mu \in m(A)$, define

$$c(\mu) := \min\{a \in A;\ \mu(a) \neq 0\}$$

and call it *the class of* μ. Define

$$A(\mu) := \{a \in A;\ a > c(\mu) \text{ or } a = c(\mu) \in A_{\overline{1}}\}.$$

LEMMA (6.5.1). *Give Ξ the restriction of the standard ordering of* (i) *of Example* (1.3.3). *If $\Xi \subset sm_r(A, I)$ is good, then, for $M \in \Xi$,*

(i) $\quad$ *Basis* $(\{ml\ t(M)\}^c) = A(\mu(M))$,

(ii) $\quad$ $ml\ t(M) = sm\ (A(\mu(M))^c)$.

Proof. Fix $M \in \Xi$ and put $\tilde{A} := A(\mu(M))$ for brevity. Put $c = c(\mu(M))$.

First we show $\tilde{A} \subset \{ml\ t(M)\}^c$.

Suppose $a > c$. If $a \in A_{\overline{1}}$ and $\mu(M)(a) > 0$, then $a \in \{\mathrm{mlt}(M)\}^c$ by definition. If $a \in A_{\overline{1}}$ and $\mu(M)(a) = 0$ or $a \in A_{\overline{0}}$, then $\delta_a + M = \delta_c + M'$ with $M' = M + \delta_a - \delta_c \in \Xi$. Since $M' < M$, we have $a \in \{ml\ t(M)\}^c$.

Suppose $a = c \in A_{\overline{1}}$. Then a is in the set $\{ml\ t(M)\}^c$ because of $\mu(M)(c) > 0$. Thus we have proved $\tilde{A} \subset \{ml\ t(M)\}^c$, which implies

$$\{ml\ t(M)\}^c \supset \rho_\infty(\tilde{A}).$$

Next we show that

(2) $\quad$ $\{ml\ t(M)\}^c \subset \rho_\infty(\tilde{A})$.

Suppose μ does not belong to $\rho_\infty(\tilde{A})$, which means

(3) $\quad$ $\mu(a) = 0$ if $a > c$ or $a = c \in A_{\overline{1}}$.

We show that $\mu + M$ does not belong to $\rho_\infty(\Xi^M)$. Then μ belongs to $ml\ t(M)$. Hence (2). Suppose the contrary: $\mu + M \in \rho_\infty(\Xi^M)$. Then we have

$$\mu + M = \nu + N$$

for some $\nu \in sm(A)$ and $N \in \Xi^M$. We shall show that this implies $\mu(N) \leqslant \mu(M)$, whence $\mu(M) = \mu(N)$ because of $|\mu(N)| = |\mu(N)|$. This is a contradiction.

Since $N < M$, we have $c(\mu(N)) \geqslant c$ and

$$\mu(M)(a) = \mu(N)(a) = 0 \quad \text{for} \quad a < c$$

and

$$\mu(M)(c) \geqslant \mu(N)(c).$$

For $a > c$, the assumption (3) implies

$$\mu(M)(a) = \nu(a) + \mu(N)(a) \geqslant \mu(N)(a).$$

Hence we have proved $\{ml\ t(M)\}^c = \rho_\infty(A(\mu(M)))$, which is equivalent to the assertion (ii). The assertion (i) follows obviously from (ii). g.e.d. ■

COROLLARY (6.5.2). *If $\Xi \subset sm_r(A, I)$ is good, then it is 1-acyclic.* ■

Let Ξ be a good subset of $sm_r(A, I)$ with respect to a linear order of A. We give to $m(A, I)$ the ordering in (ii) of Example (1.3.3). Let $\mathscr{E} = \mathscr{E}(\varphi)$ with $\varphi \in F^0(J_\infty, \Xi)$. The next lemma gives an explicit description of the subset $\bar{\rho}_1(\mathscr{E})$ of $\rho_1(\mathscr{E})$ defined in the previous section when Ξ is good.

LEMMA (6.5.3).

$$\bar{\rho}_1(\mathscr{E}) = \mathscr{E} \cup \{d_a \psi_M;\ M \in \Xi,\ a \in A(\mu(M))^c\}.$$

Proof. By Lemma (6.5.1), $ml\ t(M) = sm(A(\mu(M))^c)$. Hence

$$\bar{\rho}_1(\Xi) = \Xi \cup \{\delta_a + M;\ a \in A(\mu(M))^c\}.$$

This implies

$$\tilde{\varphi}_{\delta_a + M} = d_a \varphi_M, \qquad \psi_{\delta_a + M} = d_a \psi_M$$

and

$$\begin{aligned}\bar{\rho}_1(\mathscr{E}) &= \{\psi_N;\ N \in \rho_1(\Xi)\}\\ &= \mathscr{E} \cup \{\psi_{\delta_a + N};\ a \in A(\mu(M))^c\}\\ &= \mathscr{E} \cup \{d_a \psi_N;\ a \in A(\mu(M))^c\}. \qquad \text{q.e.d.}\end{aligned}$$ ■

Now we can rephrase the compatibility of regular systems of super differential equations with good principal index set in a style close to the classical involutivity.

PROPOSITION (6.5.4). *Suppose* Ξ *is a good subset of* $sm_r(A, I)$. *Let* $\mathscr{E} = \mathscr{E}(\varphi)$ *with* $\varphi \in F^0(J_\infty, \Xi)$. *Then* $\mathscr{E}$ *is involutive if and only if for all* $M \in \Xi$ *and* $a \in$ $\in A(\mu(M))$,

$$d_a \psi_M \equiv 0 \qquad (\text{mod } \bar{p}_1(\mathscr{E})).$$

Proof. Obvious from Proposition (6.4.2) and Lemma (6.5.3). q.e.d. ■

REMARK (6.5.5). When $\Xi \subset sm_r(A_{\bar{0}}, I)$ or $\Xi \subset sm_r(A_{\bar{1}}, I)$ it is possible at least «locally» to make a linear change of coordinates so that a regular system $\mathscr{E}$ with $pi(\mathscr{E}) = \Xi$ turns out to have a good set of principal indices. This statement does not seem true for general Ξ.

§7. SYMBOLS

We fix as before $\mathbb{Z}_2$-sets A and I and an augmented superalgebra (G, ϵ).

7.1. Involutive submodules

Let $E = \oplus_{i \in I} \mathbb{R} \cdot e_I$ be the $\mathbb{Z}_2$-graded $\mathbb{R}$-vector space with the parity $\tilde{e}_i = \tilde{i}$. For a superalgebra F, the tensor product

$$\mathbb{R}[A] \otimes_{\mathbb{R}} E \otimes_{\mathbb{R}} F = \oplus_{i \in \mathbb{Z}} \mathbb{R}_i[A] \otimes_{\mathbb{R}} E \otimes_{\mathbb{R}} F$$

will be considered as a $\mathbb{Z}$-graded supermodule over the $\mathbb{Z}$-graded superalgebra $\mathbb{R}[A] \otimes F := \oplus_{i \in \mathbb{Z}} \mathbb{R}_i[A] \otimes F$. We put $w_a := Z_a \otimes 1 \in \mathbb{R}[A] \otimes F$ $(a \in A)$ and $e_M := w^\mu e_i$ $(M = (\mu, i) \in sm(A, I))$. As an F-module

$$\mathbb{R}[A] \otimes E \otimes F = \oplus_{M \in sm(A, I)} F \cdot e_M .$$

We assume that $sm(A, I)$ is endowed with an admissible order « $<$ », and use the notations of §2.3 for the F-module $\mathbb{R}[A] \otimes E \otimes F$.

Let S be a regular F-submodule of $\mathbb{R}[A] \otimes E \otimes F$ and Ξ its principal index set. We call S *involutive* (with respect to the order « $<$ ») if

(i) Ξ is 1-acyclic,

and

(ii) $pt((\mathbb{R}_1[A] \otimes F) \cdot S) = (\mathbb{R}_1[A] \otimes F) \cdot pt(S)$.

We note that generally the right hand side of (ii) is a proper subset of the left hand side, as the following example shows.

Example (7.1.1). Let $F = \mathbb{R}$, $A = A_{\bar{0}} = \{x, y\}$, with the order $x > y$ and

$I = I_{\overline{0}} = \{u\}$. Identify $sm(A, I)$ with $sm(A) = m(A)$ and give it the order in (i) of Example (1.3.3). Let

$$S = \mathbb{R} \cdot (x^2 + y^2) \oplus \mathbb{R} \cdot xy \subset \mathbb{R}_2[A].$$

Then $pi(S) = \{x^2, xy\}$ is good and hence 1-acyclic. Further

$$pt(S) = \mathbb{R} \cdot x^2 + \mathbb{R} \cdot xy$$

and

$$(\mathbb{R}_1[A]) \cdot pt(S) = \mathbb{R} \cdot x^3 + \mathbb{R} \cdot x^2y + \mathbb{R} \cdot xy^2.$$

But

$$\mathbb{R}_1[A] \cdot S = \{x^3, \ x^2y, xy^2, y^3\} = pt(\mathbb{R}_1[A] \cdot S),$$

whence the left hand side of (ii) properly includes the right hand side in this case. ∎

REMARK (7.1.2). (i) When $A = A_{\overline{0}}$ and $I = I_{\overline{0}}$, an F-submodule S of $F \otimes \mathbb{R}[A] \otimes E$ is usually called involutive if the Koszul complex associated with the $F[A]$-module S is acyclic (cf. [5] for example). It is not difficult to see that if S is involutive in the usual sense, then, by a suitable linear change of coordinates, the principal index set $pi(S) \subset sm_r(A, I)$ is good and hence 1-acyclic and the condition (ii) is also satisfied, i.e., S is involutive in our sense. Conversely it is easy to prove that if S is involutive in our sense then it is involutive also in the usual homological sense.

(ii) It can be shown that if S is involutive then

$$pt(\mathbb{R}_r[A] \otimes F) \cdot S) = (\mathbb{R}_r[A] \otimes F) \cdot pt(S)$$

for all r. Note incidently that this means that if $\{s_\lambda; \lambda \in \Lambda\}$ is an F-basis of S such that $\{pi(s_\lambda); \lambda \in \Lambda\} = pi(S)$, then $\{s_\lambda; \lambda \in \Lambda\}$ is a Gröbner basis of the sub $\mathbb{R}[A] \otimes F$-module of $\mathbb{R}[A] \otimes F \otimes E$ generated by S. ∎

7.2. Symbol modules of systems of super differential equations

We use the notations of §2.2 and §7.1. For $r \in \mathbb{Z}$ we define

$$\sigma_r : \mathscr{F}_r \to \mathbb{R}_r[A] \otimes E \otimes \mathscr{F}_r$$

by

$$\sigma_r(F) = \Sigma_{M \in sm_r(A, I)} e_M \otimes \partial_M F.$$

Example (7.2.1). Let $A_{\overline{0}} = \{x, y\}$, $A_{\overline{1}} = \{\theta, \eta\}$, $I = I_{\overline{0}} = \{u\}$ and identify

$\mathbb{R}[A] \otimes E$ with $\mathbb{R}[A]$ by $f \otimes u \leftrightarrow f \ (f \in \mathbb{R}[A])$. Then

$$\sigma_2(\theta u_{xy} u^2_{\theta\eta} + u_{x\theta} u_{y\eta} u)$$
$$= xy \otimes \theta u^2_{\theta\eta} + 2\theta\eta \otimes \theta\, u_{xy} u_\theta + x\theta \otimes \theta u u_{y\eta} - y\eta \otimes \theta u u_{x\theta}. \qquad \blacksquare$$

Obviously we have

LEMMA (7.2.2).

(i) $\quad Ker\, \sigma_r = \mathcal{F}_{r-1}$,

(ii) $\quad \sigma_r(u_w) = \zeta^w e_i$, for $w = (w, i) \in w_r(A) \times I$,

(iii) $\quad \sigma_r(FG) = \sigma_r(F)G + F\sigma_r(G) \qquad$ for $\quad F, G \in \mathcal{F}_r$. $\blacksquare$

The following lemma gives the relation between σ_{r+s} and σ_r.

LEMMA (7.2.3).

$$\sigma_{r+s}(d^\mu F) = \zeta^\mu \sigma_r(F) \text{ for } \mu \in sm_r(A) \text{ and } F \in \mathcal{F}_r .$$

Proof. We may assume $\mu = \delta_a$.

$$\sigma_{r+1}(d_a F) = \sigma_{r+1}(\Sigma_{(\mu,i) \in sm(A,I)}\ u_{(aw(\mu),i)}\ \partial_{(\mu,i)} F)$$
$$= \Sigma_{M \in sm_r(A,I)} \zeta_a e_M \otimes \partial_M F = \zeta_a \sigma_r(F). \qquad \text{q.e.d.} \qquad \blacksquare$$

Let $\mathcal{E} \subset \mathcal{F}_r$ be a system of super differential equations. Let $\bar{\bar{\mathcal{F}}}_r$ be the quotient algebra of $\mathcal{F}_r$ by the ideal generated by $\mathcal{E}$. Let $\tau : \mathcal{F}_r \to \bar{\bar{\mathcal{F}}}_r$ be the projection. Put

$$\bar{\sigma}_r := (1 \otimes \tau) \circ \sigma_r : \mathcal{F}_r \to \mathbb{R}_r[A] \otimes E \otimes \bar{\bar{\mathcal{F}}}_r.$$

Then

$$\bar{\sigma}_r(hg) = \bar{\sigma}_r(h)\, \tau(g)$$

for $h \in \mathcal{F}_r \cdot \mathcal{E}$ and $g \in \mathcal{F}_r$, whence

$$\mathrm{Symb}_r(\mathcal{E}) := \bar{\sigma}_r(\mathcal{F}_r \cdot \mathcal{E})$$

is an $\bar{\bar{\mathcal{F}}}_r$-submodule of $\mathbb{R}_r[A] \otimes E \otimes \bar{\bar{\mathcal{F}}}_r$, which depends only on $\mathcal{F}_r \cdot \mathcal{E}$. This is called *the symbol module of* $\mathcal{E}$.

7.3. Condition of involutiveness in terms of symbol modules

Fix an admissible linear order on $sm(A, I)$. Let $\mathcal{E}$ be a regular system and put $\Xi = pi(\mathcal{E}) \subset sm_r(A, I)$.

THEOREM (7.3.1). *A system of super differential equations $\mathscr{E}$ is involutive if and only if the following two conditions are satisfied*:

(i) $\quad Symb(\mathscr{E})$ *is involutive,*

(ii) $\quad \mathscr{F}_{r+1} \cdot \rho_1(\mathscr{E}) \cap \mathscr{F}_r = \mathscr{F}_r \cdot \mathscr{E}$.

Proof. Let $\mathscr{E} = \mathscr{E}(\varphi)$ for some $\varphi \in F^0(J_r, \Xi)$.

Suppose $\mathscr{E}$ is involutive. By Proposition (6.4.2) we have for $(\delta_a, M) \in$ Basis $(\mathrm{mlt}(A, \Xi)^c$

$$(5) \qquad d_a \psi_M \in \mathscr{F}_{r+1} \cdot \overline{\rho}_1(\mathscr{E}),$$

(cf. §6.4 for the notation $\overline{\rho}_1(\mathscr{E})$), which implies

$$(6) \qquad d_a \psi_M = \sum_{N \in \rho_1(\Xi)} C_N \psi_N$$

with $C_N \in \mathscr{F}_{r+1}$ and $C_N \in \mathscr{F}_r$ for $N \in sm_{r+1}(A, I)$, by virtue of Lemma (7.3.2) below.

Put

$$\rho_1^0(\Xi) := \rho_1(\Xi) \cap sm_{r+1}(A, I).$$

LEMMA (7.3.2). *Suppose*

$$\psi = C_0 + \Sigma_{N \in sm_{r+1}(A,I)} C_N Z_N$$

with $C_0, C_N \in \mathscr{F}_r$ belongs to $\mathscr{F}_{r+1} \cdot \overline{\rho}_1(\mathscr{E})$. Then

$$\psi = \Sigma_{N \in \rho_1(\Xi)} \tilde{C}_N \psi_N,$$

with

$$\tilde{C}_N \in \mathscr{F}_r \ \textit{for} \ N \in \rho_1^0(\Xi)$$

and

$$\tilde{C}_N \in \mathscr{F}_r + \Sigma_{M \in sm_{r+1}(A,I)} \mathscr{F}_r \cdot Z_M \quad \textit{for} \quad N \in \Xi.$$

Proof. Obviously, modulo an element in $\Sigma_{N \in \rho_1^0(\Xi)} \mathscr{F}_r \cdot \psi_N$, we have

$$\psi \equiv \psi_0 := C_0' + \Sigma_{N \in sm_{r+1}(A,I) \setminus \rho_1(\Xi)} C_N' Z_N$$

with $C_N', C_0' \in \mathscr{F}_r$. By Proposition (3.4.4), there exists $\tilde{\varphi} \in F^{00}(J_1, \rho_1(\Xi))$ such that $\mathscr{E}(\overline{\rho}_1(\varphi)) \sim \mathscr{E}(\tilde{\varphi})$. Make now the substitutions

$$Z_M = \tilde{\varphi}_M, \quad \text{for} \quad M \in \rho_1(\Xi).$$

Since $\psi_0 \in \mathscr{E}(\bar{\rho}_1(\varphi))$, we have by Lemma (3.4.2),

$$\psi_0 = C_0'' + \Sigma_N C_N'' Z_N,$$

where $C_i'' = C_i'(Z_M = \tilde{\varphi}_M; M \in \Xi)$. Since 1 and $\{Z_N; N \in sm_{r+1}(A, I)\}$ are linearly independent over $\mathscr{F}_r$, we obtain C_0'', $C_N'' = 0$, whence C_0', $C_N' \in \mathscr{F}_r \cdot \mathscr{E}$. Thus

$$\psi_0 = \Sigma_{M \in \Xi}(C'_{0M}\psi_M + \Sigma_N C'_{NM}\psi_M Z_N) = \Sigma_{M \in \Xi} \; C_M \psi_M.$$

q.e.d. ∎

Applying σ_{r+1} to (5), we obtain

$$\zeta_a \sigma_r(\psi_M) = \Sigma_{N \in \rho_1^0(\Xi)} C_N \sigma_{r+1}(\psi_N) + \Sigma_{N \in \Xi} \sigma_{r+1}(C_N)\psi_N,$$

where we note that $\sigma_{r+1}(C_N) = 0$ for $N \in \rho_1^0(\Xi)$. For $N \in \rho_1^0(\Xi)$, there exists by definition a unique $(\delta_b, L) \in ml\ t(A, \Xi)$ such that $\delta_b + L = N$ and $\psi_N = d_b \psi_L$. Hence

$$\zeta_a \sigma_r(\psi_M) = \Sigma_{(\delta_b, L) \in ml\,t(A, \Xi)} C_N \zeta_b \sigma_r(\psi_L) + \Sigma_{N \in \Xi} \sigma_{r+1}(C_N)\psi_N.$$

Considering in $\mathbb{R}[A] \otimes E \otimes \mathscr{F}_{r+1} / \mathscr{F}_{r+1} \cdot \mathscr{E}$, we obtain

$$\zeta_a \bar{\sigma}_r(\psi_M) = \Sigma_{(\delta_b, L) \in ml\,t(A, \Xi)} \bar{C}_N \zeta_b \bar{\sigma}_r(\psi_L).$$

This is an identity in $\mathbb{R}[A] \otimes E \otimes \bar{\bar{\mathscr{F}}}_r$ and implies

$$\mathbb{R}_1[A] \cdot Symb(\mathscr{E}) = \bigoplus_{(\delta_b, L) \in ml\,t(A, \Xi)} \bar{\bar{\mathscr{F}}}_r \cdot \zeta_b \bar{\sigma}_r(\psi_L),$$

whence

$$pi(\mathbb{R}_1[A] \cdot Symb(\mathscr{E})) = sm_1(A, 1I) \boxplus \Xi,$$

which means $Symb(\mathscr{E})$ is involutive.

As to (ii), the Corollary (5.2.7) implies

$$\mathscr{F}_r \cap \mathscr{F}_{r+1} \cdot \rho_1(\mathscr{E}) \subset \mathscr{F}_r \cdot \{Z_N - \varphi_N; N \in \rho_1(\mathscr{E}) \cap J_r = \Xi\}$$
$$= \mathscr{F}_r \cdot \mathscr{E},$$

whence (ii) is true.

Conversely suppose (i) and (ii) are true. Since

$$Symb(\mathscr{E}) = \oplus_{M \in \Xi} \bar{\bar{\mathscr{F}}}_r \cdot b_M \qquad (b_M = \bar{\sigma}_r(\psi_M)),$$

(i) is equivalent to

$$\mathbb{R}_1[A] \cdot Symb(\mathscr{E}) = \oplus_{N \in \rho_1^0(\mathscr{E})} \bar{\bar{\mathscr{F}}}_r \cdot b_N,$$

where $b_N = \zeta_b b_L$ for $N = 1_b + L$ with $(1_b, L) \in ml\ t(A, \Xi)$. Hence for $(a, M) \in Basis(ml\ t(A, I)^c)$,

(7) $$\zeta_a b_M = \Sigma_{N \in \rho_1^\circ(\Xi)} \overline{C}_N b_N .$$

Choose $C_N \in \mathscr{F}_r$ which represents $\overline{C}_N$. Then

$$\sigma_{r+1}(d_a \psi_M - \Sigma_{N \in \rho_1^\circ(\Xi)} C_N \psi_N)$$
$$= \zeta_a \sigma_r(\psi_M) - \Sigma_{(\delta_b, L) \in mlt(A,I)} C_{\delta_b + L} \quad \zeta_b \sigma_r(\psi_L).$$

Since the projection from $\mathbb{R}[A] \otimes E \otimes \mathscr{F}_{r+1}$ to $\mathbb{R}[A] \otimes E \otimes \overline{\mathscr{F}}_{r+1}$ annihilates the left hand side by (7), we have,

$$d_a \psi_M - \Sigma_{N \in \rho_1^\circ(\Xi)} C_N \psi_N = \Sigma_{M \in sm_{r+1}(A,I)} C_M Z_M + C_0$$

with $C_M \in \mathscr{F}_r \cdot \mathscr{E}$, $C_0 \in \mathscr{F}_r$. Since

$$C_0 \in \mathscr{F}_r \cap \mathscr{F}_{r+1} \cdot \rho_1(\mathscr{E}),$$

we have $C_0 \in \mathscr{F}_r \cdot \mathscr{E}$. Hence

$$d_a \psi_M \in \mathscr{F}_{r+1} \cdot \overline{\rho}_1(\mathscr{E}).$$

Hence $\mathscr{E}$ is involutive. q.e.d. ∎

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